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An elementary existence theorem for entire functions

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It is proved that, for any m given distinct real numbers $a_1, \ldots, a_m$, there exist transcendental entire functions $f(z)$ at most of order m for which all the values

$$f^{(n)}\left(a_k\right) \quad \begin{pmatrix} n = 0, 1, 2, \ldots \\ k = 1, 2, \ldots, m \end{pmatrix}$$

are rational integers.

1.

Let $a_1, \ldots, a_m$, where $m \geq 2$ (the case $m = 1$ is trivial), be finitely many given distinct real numbers, and let

$$\alpha_{hj} \quad \begin{pmatrix} h = 0, 1, 2, \ldots \\ j = 1, 2, \ldots, m \end{pmatrix}$$

be infinitely many real numbers still to be selected. Put

$$g(z) = \left(z-a_1\right) \ldots \left(z-a_m\right) , \quad A_k = \left|g'\left(a_k\right)\right| = \prod_{\substack{j=1 \\ j \neq k}}^{m} \left|a_k-a_j\right| ,$$

so that all A_k are positive numbers. Let further

$$g_{hj}(z) = \frac{\alpha_{hj}}{z-a_j} \cdot \frac{g(z)^{h+1}}{h!(h+1)!^{m-1}} \quad \begin{pmatrix} h = 0, 1, 2, \ldots \\ j = 1, 2, \ldots, m \end{pmatrix}$$

and

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$$f(z) = \sum_{h=0}^{\infty} \sum_{j=1}^{m} g_{hj}(z) .$$

Then, for all non-negative integers n,

$$g_{hj}^{(n)}(a_k) = 0 \text{ if } j = k \text{ and } h > n \text{, or if } j \neq k \text{ and } h \geq n ,$$

but

$$g_{nk}^{(n)}(a_k) = a_{nk} \prod_{\substack{j=1 \\ j\neq k}}^{m} \left\{ \frac{(a_k - a_j)^{n+1}}{(n+1)!} \right\} = \mp \frac{a_{nk} \cdot A_k^{n+1}}{(n+1)!^{m-1}} .$$

It follows therefore that

$$(1) \quad f^{(n)}(a_k) = \mp \frac{a_{nk} A_k^{n+1}}{(n+1)!^{m-1}} + \sum_{h=0}^{n-1} \sum_{j=1}^{m} g_{hj}^{(n)}(a_k) \quad \left(\begin{matrix} n = 0, 1, 2, \ldots \\ k = 1, 2, \ldots, m \end{matrix} \right) .$$

2.

Here, in the double sum on the right-hand side, there occur only coefficients a_{hj} with $0 \leq h \leq n-1$. This basic equation (1) enables us therefore to select the coefficients a_{hj} suitably by induction on h, as follows.

Firstly, take

$$a_{0k} = \mp A_k^{-1} \quad (k = 1, 2, \ldots, m) ,$$

so that

$$f(a_k) = \mp 1 \quad (k = 1, 2, \ldots, m) .$$

Secondly, let $n \geq 1$, and assume that all coefficients a_{hj} with $0 \leq h \leq n-1$ have already been fixed. There exist then, for each suffix $k = 1, 2, \ldots, m$, just two real values of a_{hj} such that simultaneously

$$-(n+1)!^{m-1} \leq a_{nk} A_k^{n+1} \leq +(n+1)!^{m-1} , \quad a_{nk} \neq 0 ,$$

and

$$f^{(n)}(a_k) \quad \text{is a rational integer.}$$

With the coefficients a_{hj} so chosen, we find for $f(z)$ the upper estimate

$$|f(z)| \leq \sum_{h=0}^{\infty} \sum_{j=1}^{m} \frac{|g(z)|}{A_j|z-a_j|} \frac{|g(z)|^h}{A_j^h . h!} ,$$

which is equivalent to

$$|f(z)| \leq \sum_{j=1}^{m} \frac{|g(z)|}{A_j|z-a_j|} \cdot \exp\left(\frac{|g(z)|}{A_j}\right) .$$

This estimate shows that the series for $f(z)$ converges absolutely and uniformly in every bounded set of the complex plane and defines an entire function of z at most of order m .

In fact, since there are always two choices for each of the coefficients a_{hj} , we obtain a non-countable set of such functions $f(z)$. Hence, amongst these functions, there are also non-countably many which are not polynomials and hence are transcendental entire functions. The following result has thus been established.

THEOREM. *Let* $a_1, \ldots, a_m$ *be finitely many distinct real numbers where* $m \geq 2$. *There exist non-countably many entire transcendental functions* $f(z)$ *at most of order* m *such that all the values*

$$f^{(n)}(a_k) \quad \begin{pmatrix} n = 0, 1, 2, \ldots \\ k = 1, 2, \ldots, m \end{pmatrix}$$

are rational integers.

3.

Two interesting questions arise now which I have not been able to solve. The first one concerns the extension of the theorem to the case of infinite sequences.

PROBLEM A. *Let* $S = \{a_k\}$ *be an infinite sequence of distinct real numbers without finite limit points. Which conditions has* S *to satisfy if there is to exist at least one entire function* $f(z)$ *not a constant* *such that all the values*

$$f^{(n)}(a_k) \quad \begin{pmatrix} n = 0, 1, 2, \ldots \\ k = 1, 2, 3, \ldots \end{pmatrix}$$

are rational integers?

In the special case when S consists of the integral multiples of a fixed positive number, I have proved that there do exist entire functions with this property; see [1].

To formulate a second problem, let again $a_1, \ldots, a_m$, $m \geq 2$, be a finite set of distinct real numbers, and let $f(z)$ be one of the functions the existence of which has been established in the theorem. Since we may replace z by $z - a_m$, there is no loss of generality in assuming that $a_m = 0$. With this choice, the set $\{a_1, \ldots, a_{m-1}\}$ has then non-countably many possibilities. On the other hand, it is easily seen that there are only countably many entire functions of the form

$$f(z) = \sum_{h=0}^{\infty} f_h \frac{z^h}{h!}$$

with rational integral coefficients f_h which satisfy algebraic differential equations. Taking $m = 2$, we arrive therefore at the following question.

PROBLEM B. *For which real values of the number* $a_1 \neq 0$ *does there exist an entire transcendental function* $f(z)$ *which*

(i) satisfies an algebraic differential equation, and

(ii) has the property that all the values

$$f^{(n)}(0) \text{ and } f^{(n)}(a_1) \quad (n = 0, 1, 2, \ldots)$$

are rational integers?

Such functions always exist when a_1 is a rational multiple of π ; but I do not know whether this is the only case.

Reference

[1] Kurt Mahler, "An arithmetic remark on entire periodic functions", *Bull. Austral. Math. Soc.* 5 (1971), 191-195.

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