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# On a class of diophantine inequalities

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Dedicated to B. Segre, on his 70th birthday, 16 February 1973.

As a special case of more general results, it is proved in this note that, if $\alpha$ is any real number and $\delta$ any positive number, then there exists a positive integer $X$ such that the inequality

$$\left|X\left(\tfrac{3}{2}\right)^h - Y_h - \alpha\right| < \delta$$

has infinitely many solutions in positive integers $h$ and $Y_h$.

The method depends on the study of infinite sequences of real linear forms in a fixed number of variables. It has relations to that used by Kronecker in the proof of his classical theorem and can be generalised.

## 1.

For real $\alpha$ put

$$\|\alpha\| = \min_{y=0,\pm1,\pm2,\ldots} |\alpha-y| ,$$

so that $\|\alpha\|$ denotes the distance of $\alpha$ from the nearest integer and hence that

$$0 \le \|\alpha\| \le \tfrac{1}{2} .$$

By $H_0$ we understand a fixed strictly increasing infinite sequence of positive integers $h$ ($H_0$ usually will be the set of all positive

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integers), and $H$ denotes some infinite subsequence of $H_0$, not necessarily always the same.

## 2.

Let $r$ be a fixed and $n$ a variable positive integer; let further $S_n$ be the set of all $r$-vectors $\mathbf{x} = (x_1, \ldots, x_r)$ with integral components satisfying

$$1 \le \max(|x_1|, \ldots, |x_r|) \le n .$$

Thus $S_n$ is a finite set, and all vectors in $S_n$ are distinct from the zero vector

$$\mathbf{o} = (0, \ldots, 0) .$$

Next consider an infinite sequence of $r$-vectors

$$\mathbf{a}_h = (a_{h1}, \ldots, a_{hr}) \quad (h \in H_0)$$

with real components and the associated linear forms

$$L_h(\mathbf{x}) = a_{h1}x_1 + \ldots + a_{hr}x_r \quad (h \in H_0)$$

in $\mathbf{x}$. Then put

$$M_h(n) = \min_{\mathbf{x} \in S_n} \|L_h(\mathbf{x})\| \quad (h \in H_0)$$

and

$$M(n) = \limsup_{\substack{h \to \infty \\ h \in H_0}} M_h(n) .$$

It is obvious that

$$0 \le M_h(n) \le \tfrac{1}{2} \quad (h \in H_0)$$

and hence that also

$$0 \le M(n) \le \tfrac{1}{2} .$$

## 3.

For $n \ge 3$ these upper bounds for $M_h(n)$ and $M(n)$ can be improved.

For this purpose, denote by $y$ a further integral variable. The system of $r + 1$ linear forms

$$n^{-1}x_1, \ldots, n^{-1}x_r, \quad n^r(a_{h1}x_1 + \ldots + a_{hr}x_r - y) \quad (h \in H_0)$$

in $x_1, \ldots, x_r$, $y$ has the determinant $-1$. Hence, by Minkowski's Theorem on linear forms, there exist integers

$$x_{h1}, \ldots, x_{hr}, y_h$$

not all zero, which in general will depend on $h$, such that simultaneously

$$\max(|x_{h1}|, \ldots, |x_{hr}|) \le n, \quad |a_{h1}x_{h1} + \ldots + a_{hr}x_{hr} - y_h| < n^{-r} \quad (h \in H_0) .$$

*Here at least one of the first* $r$ *integers*

$$x_{h1}, \ldots, x_{hr}$$

*does not vanish.* For otherwise $y_h \neq 0$, whence

$$1 \le |y_h| < n^{-r} \le 1 ,$$

which is impossible.

The vector

$$\mathsf{x}_h = (x_{h1}, \ldots, x_{hr})$$

therefore lies in $S_n$ and in addition satisfies the inequality

$$\|L_h(\mathsf{x}_h)\| < n^{-r} \quad (h \in H_0)$$

From this it follows immediately that

(1) $$0 \le M_h(n) < n^{-r} \quad (h \in H_0)$$

and hence also that

(2) $$0 \le M(n) \le n^{-r} .$$

On the other hand, since obviously $S_n \subset S_{n+1}$, it is clear that

$$M_h(1) \ge M_h(2) \ge M_h(3) \ge \ldots \ge 0 \quad (h \in H_0) ,$$

from which it is easily deduced that also

$$M(1) \geq M(2) \geq M(3) \geq \ldots \geq 0 .$$

## 4.

The definition of $M(n)$ as an upper limit implies that there exists a subsequence $H$ of $H_0$ such that

$$\lim_{\substack{h\to\infty \\ h\in H}} M_h(n) = M(n) .$$

Here, to each suffix $h$ in $H$, we can find a vector $\mathrm{x}_h$ in $S_n$ such that

$$M_h(n) = \|L_h(\mathrm{x}_h)\| \quad (h \in H) ;$$

note that $\mathrm{x}_h$ need not be the same as the vector $\mathrm{x}_h$ constructed in §3.

As $h$ runs over $H$, $\mathrm{x}_h$ is restricted by the condition of belonging to the finite set $S_n$. Therefore, if necessary, $H$ can be replaced by an infinite subsequence which we call again $H$ such that, without loss of generality,

$$\mathrm{x} = \mathrm{x}_h \quad \text{for all} \quad h \in H$$

is a fixed vector in $S_n$ independent of $h$; naturally,

$$\mathrm{x} \neq 0 .$$

Since this vector has the basic property that

$$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(\mathrm{x})\| = M(n) , \tag{3}$$

the following result has been established.

**LEMMA 1.** *For every positive integer* $n$ *there exist an infinite subsequence* $H$ *of* $H_0$ *and a constant vector* $\mathrm{x}$ *in* $S_n$ *with the property* (3).

## 5.

In this lemma, $H$ will in general be a proper subsequence of $H_0$ as the following example shows.

Fix $n$ and choose $r = 1$ so that $\mathbf{a}_h$ and $\mathsf{x}$ are now scalars $a_h$ and $x$. As the linear forms take

$$L_h(x) = \begin{cases} x & \text{if } h \text{ is even,} \\ x\sqrt{2} & \text{if } h \text{ is odd.} \end{cases}$$

In this example, $M_h(n)$ evidently vanishes for even $h$ (we may put $x = 1$), but is positive and independent of $h$ for odd $h$. Hence also $M(n)$ is positive. Thus, if $H_0$ is the set of all positive integers $h$, $H$ in (3) essentially (that is, except for possibly finitely many even numbers) is the sequence of all odd integers.

## 6.

Consider again the general case, but assume that, for a certain $n$, $M(n) = 0$. Since $M_h(n) \geq 0$ for all $h \in H_0$, it is clear that now the *upper limit* in the definition of $M(n)$ becomes the *limit*, hence that (3) takes the form

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(\mathsf{x})\| = 0 . \tag{4}$$

Denote by $\alpha$ an arbitrary real number which is not an integer. The relation (4) implies that

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(\mathsf{x})-\alpha\| = \|\alpha\| > 0 .$$

This formula suggests the problem whether there exist an infinite subsequence $H$ of $H_0$ and an integral vector $\mathsf{X}$ distinct from $\mathsf{x}$ such that

$$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(\mathsf{X})-\alpha\| = 0 .$$

The answer to this problem depends very much on the special forms $L_h$ and the sequences $H_0$ and $H$ .

A positive answer can be given in the following trivial example. Let $r = 1$ and $n = 2$ ; let $H_0$ and $H$ be the sequences of all positive integers and of all odd positive integers, respectively; and let further

$$L_h(x) = \tfrac{1}{2}x \quad \text{for} \quad h \in H_0 .$$

Since $L_h(2) = 1$ , evidently

$$M_h(2) = M(2) = 0 .$$

On the other hand,

$$\|L_h(1)-\tfrac{1}{2}\| = 0 \quad \text{for all} \quad h \in H .$$

A negative answer holds in the following rather more interesting exámple. Let again $r = 1$ , and let $H_0$ be again the sequence of all positive integers. Assume that the forms $L_h$ have the property

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(1)\| = 0 . \tag{5}$$

Then obviously also

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(x)\| = 0 \quad \textit{for every integer} \quad x , \tag{6}$$

and hence there cannot exist a subsequence $H$ of $H_0$ and an integer $X$ satisfying

$$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(X)-\alpha\| = 0 \tag{7}$$

unless $\alpha$ is an integer.

## 7.

A simple example in which the condition (5) is satisfied and therefore also the conclusion about (7) is given by the linear forms

$$L_h(x) = h!ex \quad \text{for} \quad h \in H_0$$

where $H_0$ still denotes the sequence of all positive integers.

Of much greater interest is, however, the sequence of forms

(8) $$L_h(x) = \lambda\theta^n x \quad \text{for} \quad h \in H_0$$

where $\theta > 1$ is a fixed *algebraic* number, and $\lambda > 0$ is a constant. A theorem due to Pisot [1] (see also Salem [2]) asserts that the limit equation

$$\lim_{h\to\infty} \|\lambda\theta^h\| = 0 ,$$

that is, the condition (5), is satisfied if and only if the following two properties hold.

*(i)* $\theta = \theta^{(1)}$ *is an algebraic integer of some degree* $m \geq 1$ *such that all its algebraic conjugates* $\theta^{(2)}, \ldots, \theta^{(m)}$ *are less than* 1 *in absolute value.*

*(ii)* $\lambda$ *lies in the algebraic number field* $Q(\theta)$ *generated by* $\theta$ .

Call $\{\theta, \lambda\}$ a *Pisot pair* whenever these two properties are satisfied. By (7), such pairs have the following further property.

*(iii) If* $\alpha$ *is any real number,* $H$ *any subsequence of* $H_0$ , *and* $X$ *any integer, then the equation*

$$\lim_{\substack{h\to\infty \\ h\in H}} \|\lambda\theta^h X-\alpha\| = 0$$

*implies that* $\alpha$ *is an integer.*

If $\{\theta, \lambda\}$ is a Pisot pair, then by (6) the forms (8) satisfy

(9) $$M(n) = 0 \quad \text{for all} \quad n \geq 1 .$$

This result has a converse. For assume that $\{\theta, \lambda\}$ is not necessarily a Pisot pair, but that (9) is true. This equation (9) is equivalent to

(10) $$\lim_{\substack{h\to\infty \\ h\in H_0}} \min_{x=\pm1,\pm2,\ldots,\pm n} \|\lambda\theta^h x\| = 0 .$$

Now for every real number $\alpha$ and for every integer $g$,

$$\|g\alpha\| \le |g|.\|\alpha\| ,$$

hence

$$\|n!\lambda\theta^h\| \le n! \min_{x=\pm1,\pm2,\ldots,\pm n} \|\lambda\theta^h x\| ,$$

because all factors $x$ are divisors of $n!$. The equation (10) implies then that

$$\lim_{h\to\infty} \|n!\lambda\theta^h\| = 0 .$$

This, however, means that $\{\theta, n!\lambda\}$ and hence also $\{\theta, \lambda\}$ are Pisot pairs. Thus the following result holds.

LEMMA 2. *Let* $\theta > 1$ *be an algebraic number and* $\lambda$ *a positive number, let again* $H_0$ *be the sequence of all positive integers, and let*

$$L_h(x) = \lambda\theta^h x \quad \text{for} \quad h \in H_0 .$$

*Then* $\{\theta, \lambda\}$ *is a Pisot pair if and only if*

$$M(n) = 0 \quad \text{for all} \quad n \ge 1 .$$

## 8.

We return to the general case of §2, but assume now that for a certain value of $n$,

$$M(n) > 0 .$$

Denote by $\mathsf{X}$ the constant vector in $S_n$ given by Lemma 1 and for which

(3) $$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(\mathsf{X})\| = M(n) .$$

It follows that there exists an infinite subsequence of $H$ which we call again $H$ such that

$$\tfrac{2}{3}M(n) < \|L_h(\mathsf{x})\| < \tfrac{4}{3}M(n) \quad \text{for all} \quad h \in H$$

In explicit form, $\mathsf{x} = (x_1, \ldots, x_r)$ , and there exists to each $h \in H$ an integer $y_h$ such that the sum

$$s_h = a_{h1}x_1 + \ldots + a_{hr}x_r - y_h$$

satisfies the equation

$$|s_h| = \|L_h(\mathsf{x})\|$$

and therefore also the inequality

(11) $$\tfrac{2}{3}M(n) < |s_h| < \tfrac{4}{3}M(n) \quad \text{for all} \quad h \in H .$$

## 9.

Next let $\alpha$ be an arbitrary real number, and let $y$ be the unique integer for which the real number

$$\beta = \alpha + y$$

satisfies the inequality

(12) $$\tfrac{2}{3} < \beta \leq \tfrac{5}{3} .$$

The integral multiples

$$s_h z \quad (z = 0, \pm 1, \pm 2, \ldots)$$

of $s_h$ form an arithmetic progression of distance $|s_h| > 0$ . By (11), every open interval of length $\tfrac{4}{3}M(n)$ contains then at least one element of this progression.

We apply this property to the open interval

$$\text{from} \quad \beta - \tfrac{2}{3}M(n) \quad \text{to} \quad \beta + \tfrac{2}{3}M(n)$$

of this length and deduce that

*for every* $h \in H$ *there exists an integer* $z_h$ *such that*

$$-\tfrac{2}{3}M(n) < s_h z_h - \beta < \tfrac{2}{3}M(n) .$$

Here $\beta \leq \frac{5}{3}$ and $M(n) \leq \frac{1}{2}$, so that by (11),

$$|z_h| < \frac{\beta + \frac{2}{3}M(n)}{\frac{2}{3}M(n)} \leq \frac{5+2M(n)}{2M(n)}$$

and therefore

(13) $$|z_h| < \frac{3}{M(n)} .$$

On the other hand, $\beta > \frac{2}{3}$, and so again by (11),

$$s_h z_h > \beta - \tfrac{2}{3}M(n) \geq \tfrac{2}{3} - \tfrac{1}{3} > 0 ,$$

whence also

$$z_h \neq 0 .$$

In this construction, $z_h$ is a function of $h \in H$ which, by (13), has only finitely many possible values. Since $H$ may, if necessary, once more be replaced by a suitable infinite subsequence, we may without loss of generality assume that

$$z_h = z \quad \text{for all} \quad h \in H$$

has a *fixed* integral value independent of $h$, where by (13) and (14)

(15) $$0 < |z| < \frac{3}{M(n)} .$$

## 10.

Put finally

$$X_1 = x_1 z, \ \ldots, \ X_r = x_r z \ , \quad Y_h = y_h z + y .$$

Then $X = (X_1, \ldots, X_r)$ is an integral $r$-vector independent of $h$ such that

(16) $$1 \leq \max(|X_1|, \ldots, |X_r|) < \frac{3n}{M(n)} ,$$

while $Y_h$ is an integer which in general depends on $h$. In this new notation, the lower and upper estimates for $s_h z_h - \beta$ take the form

$$-\tfrac{2}{3}M(n) < L_h(X) - Y_h - \alpha < \tfrac{2}{3}M(n) \quad \text{for all} \quad h \in H .$$

Since $\frac{2}{3}M(n) < \frac{1}{2}$ , this is equivalent to

(17) $$\|L_h(X)-\alpha\| < \tfrac{2}{3}M(n) \quad \text{for all} \quad h \in H .$$

Thus the following result has been obtained.

**LEMMA 3.** *For a certain* $n \geq 1$ *let* $M(n) > 0$ . *Then, to every real number* $\alpha$ , *there exist an infinite subsequence* $H$ *of* $H_0$ *and a constant integral vector* $X$ *such that both* (16) *and* (17) *are satisfied.*

This lemma becomes particularly interesting when $M(n)$ is positive for *all* positive integers $n$ . For, by the earlier estimate (2),

$$\lim_{n\to\infty} M(n) = 0 .$$

Therefore, for sufficiently large $n$ , the right-hand side of (17) can be made arbitrarily small, giving the following result.

**THEOREM 1.** *Let* $r \geq 1$ *be a fixed integer, and let* $H_0$ *be a strictly increasing infinite sequence of positive integers. Associate with each* $h$ *in* $H_0$ *a real linear form*

$$L_h(X) = a_{h1}x_1 + \ldots + a_{hr}x_r ,$$

*and assume that the upper limit* $M(n)$ , *as defined in* §2, *is positive for every positive integer* $n$ .

*Then, given any real number* $\alpha$ *and any positive number* $\delta$ , *there exist an infinite subsequence* $H$ *of* $H_0$ *and an integral vector* $X \neq 0$ *independent of* $h$ *such that*

$$\|L_h(X)-\alpha\| < \delta \quad \textit{for all suffices} \quad h \quad \textit{in} \quad H .$$

## 11.

We combine this theorem with Lemma 2, taking $r = 1$ . Let $\theta$ and $\lambda$ be as in Lemma 2, but assume that $\{\theta, \lambda\}$ is *not* a Pisot pair. Then $M(n)$ is positive for all $n \geq 1$ , and Theorem 1 gives the following consequence.

**THEOREM 2.** *Let* $\theta > 1$ *be an algebraic number, and* $\lambda > 0$ *a* *constant. Assume that at least one of the following two properties is not satisfied.*

*(i)* $\theta = \theta^{(1)}$ *is an algebraic integer of degree* $m \geq 1$ *such that all its algebraic conjugates* $\theta^{(2)}, \ldots, \theta^{(m)}$ *have absolute values less than* 1 .

*(ii)* $\lambda$ *lies in the algebraic number field* $\mathbf{Q}(\theta)$ *generated by* $\theta$ .

*Then, given any real number* $\alpha$ *and any positive number* $\delta$ *, there exists a positive integer* $X$ *such that the inequality*

$$\|X\lambda\theta^h - \alpha\| < \delta$$

*has infinitely many solutions in positive integers* $h$ .

By way of example, this theorem can be applied to each of the inequalities

$$\left\|X\sqrt{2}\left(\frac{1+\sqrt{5}}{2}\right)^h - \alpha\right\| < \delta , \quad \|Xe(1+\sqrt{2})^h - \alpha\| < \delta , \quad \|X\lambda\left(\tfrac{3}{2}\right)^h - \alpha\| < \delta ,$$

where in the last inequality $\lambda$ may be an arbitrary positive number.

## 12.

We conclude this note with an application of Theorem 1 when $r$ is an arbitrary positive integer. For this purpose, assume that

$$L_h(x) = a_1x_1 + \ldots + a_rx_r$$

does not depend on $h$ . Any relation $M(n) = 0$ where $n \geq 1$ now implies that the numbers

$$a_1, \ldots, a_r, 1$$

are linearly dependent over the rational field $\mathbf{Q}$ . Conversely, if these numbers are linearly independent over $\mathbf{Q}$ , then $M(n)$ is positive for all $n > 1$ . In this case it follows from Theorem 1 that for every real number $\alpha$ and for every positive number $\delta$ there exist $r$ integers $X_1, \ldots, X_r$ not all zero such that

$$\|a_1X_1 + \ldots + a_rX_r - \alpha\| < \delta .$$

We obtain thus a rather special case of Kronecker's Theorem.

## References

[1] Charles Pisot, "La repartition modulo 1 et les nombres algébriques", *Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Math.* (2) **7** (1938), 205-248.

[2] Raphael Salem, *Algebraic numbers and Fourier analysis* (D.C. Heath and Co., Boston, Massachusetts, 1963).

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