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On a paper by A. Baker on the approximation of rational powers of e

by

KURT MAHLER (Canberra)

In an important paper of 1965 (Canadian Journal of Mathematics, 17, pp. 616–626), A. Baker for the first time established lower bounds for products of the form

$$\text{(I)} \qquad |x_1x_2 \ldots x_k(x_1E_1+x_2E_2+\ldots+x_kE_k)| \quad \text{and} \quad y_k|y_kE_1-y_1| \ldots |y_kE_{k-1}-y_{k-1}|.$$

Here $E_1, E_2, \ldots, E_k$ are distinct rational powers of e, with $E_k = 1$ in the second expression; the x's are distinct integers not zero, while the y's are integers where $y_k > 0$, and $k \geqslant 2$. These lower bounds involve positive constants depending only on k and the E's and are not given explicitly. The method depends on an ingenious generalization of that by C. L. Siegel in his classical paper in the Abhandlungen der Preussischen Akademie der Wissenschaften of 1929, No. 1.

I try in the present paper to carry Baker's investigations a little further by establishing lower bounds for the expressions (I) which are completely explicit and do not involve any unknown constants; the results are contained in the Theorems 1 and 2 and their corollaries. It is highly probable that better estimates can be proved if explicit formulae for Baker's approximation polynomials are used. Such formulae have been obtained recently by A. van der Poorten at the University of New South Wales.

1. This paper makes use of the following well known theorem.

LEMMA 1. *Let*

$$(g_{ij}) \qquad (i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N),$$

where $M < N$, be a matrix of integers, and let

$$G_i = \sum_{j=1}^{N} |g_{ij}| \qquad (i = 1, 2, \ldots, M).$$

Then there exist integers $x_1, \ldots, x_N$ not all zero such that

$$\sum_{j=1}^{N} g_{ij} x_j = 0 \quad \textit{for} \quad i = 1, 2, \ldots, M;$$

$$\max(|x_1|, \ldots, |x_N|) \leqslant (G_1 \ldots G_M)^{1/(N-M)}.$$

Proof. Put

$$G = [(G_1 \ldots G_M)^{1/(N-M)}],$$

where $[s]$ as usual denotes the integral part of s. There are then $(G+1)^N$ distinct vectors $\boldsymbol{x} = (x_1, \ldots, x_N)$ with integral components $x_1, \ldots, x_N$ satisfying

$$0 \leqslant x_j \leqslant G \quad (j = 1, 2, \ldots, N).$$

With each such vector $\boldsymbol{x}$ associate a second integral vector $\boldsymbol{y} = (y_1, \ldots, y_M)$ where

$$y_i = \sum_{j=1}^{N} g_{ij} x_j \quad (i = 1, 2, \ldots, M).$$

Further define for each suffix $i = 1, 2, \ldots, M$ two non-negative integers n_i and p_i by

$$n_i = \sum_{\substack{j=1 \\ g_{ij}<0}}^{N} |g_{ij}|, \quad p_i = \sum_{\substack{j=1 \\ g_{ij}>0}}^{N} |g_{ij}| \quad (i = 1, 2, \ldots, M).$$

Then evidently

$$G_i = n_i + p_i \quad (i = 1, 2, \ldots, M)$$

and for all vectors $\boldsymbol{y}$,

$$-n_i G \leqslant y_i \leqslant +p_i G \quad (i = 1, 2, \ldots, M).$$

This means that each component y_i has at most $n_i G + p_i G + 1 = G_i G + {} + 1$ possibilities, hence that the vector $\boldsymbol{y}$ has at most $(G_1 G + 1) \ldots (G_M G + 1)$ possibilities. But

$$(G+1)^N = (G+1)^M (G+1)^{N-M} > (G+1)^M G_1 \ldots G_M$$
$$\geqslant (G_1 G + 1) \ldots (G_M G + 1).$$

Hence there are more distinct vectors $\boldsymbol{x}$ than there are distinct vectors $\boldsymbol{y}$. It follows that a certain pair of distinct $\boldsymbol{x}$-vectors, $\boldsymbol{x}'$ and $\boldsymbol{x}''$ say, generate the same vector $\boldsymbol{y}$. This implies that their difference $\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}''$ is not itself the zero vector, but generates the zero vector $\boldsymbol{y} = (0, \ldots, 0)$. Since the components $x_1, \ldots, x_N$ of $\boldsymbol{x}$ evidently lie between $-G$ and $+G$, the vector $\boldsymbol{x}$ has the asserted properties.

2. Let $a_1, \ldots, a_k$, where $k \geqslant 2$, be finitely many distinct integers, and let a be a positive integer satisfying

$$(a, a_1, \ldots, a_k) = 1;$$

let further

$$A = \max(|a_1|, \ldots, |a_k|) \quad \text{and} \quad B = A + a,$$

so that

$$A \geqslant 1, \quad B \geqslant 2.$$

Put

$$E_1 = e^{a_1/a}, \ \ldots, \ E_k = e^{a_k/a}.$$

Then $E_1, \ldots, E_k$ are distinct positive numbers, and hence *the exponential functions*

$$E_1^z, \ldots, E_k^z$$

are linearly independent over the field of rational functions of z.

Next denote by $r_1, \ldots, r_k$, R variable positive integers, and put

$$r = \max(r_1, \ldots, r_k), \quad r_0 = \min(r_1, \ldots, r_k),$$

$$m = r_1 + \ldots + r_k + k - R, \quad n = r_1 + \ldots + r_k + k = m + R.$$

It will be assumed that

$$k \leqslant R \leqslant r_1 + \ldots + r_k + k - 1, \quad \text{hence that} \quad 1 \leqslant m \leqslant r_1 + \ldots + r_k \leqslant kr.$$

Since the following three expressions will occur frequently, the following abbreviations will be used,

$$k^* = \frac{k(k-1)}{2}, \quad m^* = \frac{m(m-1)}{2}, \quad R^* = \frac{1}{R}.$$

3. With each pair of suffices (i, j) satisfying $1 \leqslant i \leqslant k$, $j \geqslant 0$ associate two coefficients p_{ij} and $p(i, j)$ related by the equation

$$p(i, j) = \frac{r!}{j!} p_{ij}.$$

Both coefficients are assumed equal to zero whenever (i, j) does not belong to the set S of all pairs (i, j) satisfying $1 \leqslant i \leqslant k$, $r - r_i \leqslant j \leqslant r$.

With these coefficients form now the k polynomials

$$P_i(z) = r! \sum_{j=0}^{r} p_{ij} \frac{z^j}{j!} = \sum_{j=0}^{r} p(i, j) z^j \quad (i = 1, 2, \ldots, k)$$

and the entire function

$$F(z) = \sum_{i=1}^{k} P_i(z) E_i^z,$$

say with the power series

$$F(z) = r! \sum_{h=0}^{\infty} f_h \frac{z^h}{h!}$$

where the coefficients f_h are defined by

$$(1) \qquad a^h f_h = \sum_{i=1}^{k} \sum_{j=0}^{h} \binom{h}{j} a_i^{h-j} a^j p_{ij} \qquad (h = 0, 1, 2, \ldots).$$

Denote by G_{h+1} the sum of the absolute values of the coefficients of all the p_{ij} in this equation. Thus

$$G_{h+1} = \sum_{i=1}^{k} \sum_{j=0}^{h} \binom{h}{j} |a_i|^{h-j} a^j = \sum_{i=1}^{k} (|a_i| + a)^h$$

and therefore

$$(2) \qquad G_{h+1} \leqslant kB^h \qquad (h = 0, 1, 2, \ldots),$$

whence, in particular,

$$(3) \qquad G_1 \ldots G_m \leqslant k^m B^{m^*}.$$

4. Apply now Lemma 1 to the system of m homogeneous linear equations

$$f_h = 0 \qquad (h = 0, 1, \ldots, m-1)$$

for the n unknowns p_{ij} for which (i, j) lies in S. In the notation of the lemma, $M = m$ and $N = n$, while the maxima G_i satisfy the inequalities (2) and (3). Since $n - m = R$, the lemma shows that

There exist integers p_{ij} not all zero, but equal to zero whenever (i, j) does not lie in S, such that

$$(4) \qquad f_h = 0 \quad \textit{for} \quad 0 \leqslant h \leqslant m-1; \qquad \max_{i,j} |p_{ij}| \leqslant (k^m B^{m^*})^{R^*}.$$

Next, in the sum defining $P_i(z)$,

$$\frac{r!}{j!} = \binom{r}{j} (r-j)!$$

where it suffices to allow j to run over the interval $r - r_i \leqslant j \leqslant r$ and therefore

$$(r-j)! \leqslant r_i!.$$

Since

$$\sum_{j=0}^{r}\binom{r}{j} = 2^r,$$

it follows then from (4) that also

$$(5)\qquad \sum_{j=0}^{r}|p(i,j)| \leqslant 2^r r_i!(k^m B^{m^*})^{R^*} \qquad (i = 1, 2, \ldots, k).$$

From their construction, *the* $p(i,j)$ *likewise are integers, and they vanish whenever* (i,j) *does not lie in* S.

From (1), (2), (4), and the definition of G_{h+1} it finally follows that

$$(6)\qquad |f_h| \leqslant k(B/a)^h(k^m B^{m^*})^{R^*} \quad \text{for} \quad h \geqslant m.$$

5. By construction, not all the polynomials $P_i(z)$ vanish identically. Denote by $i_1, \ldots, i_K$, where $1 \leqslant K \leqslant k$, all the distinct suffices i for which $P_i(z) \not\equiv 0$. Then, by what was said in § 2 about the exponential functions $E_1^z, \ldots, E_k^z$, *the* K *functions*

$$g_1(z) = P_{i_1}(z)E_{i_1}^z, \quad \ldots, \quad g_K(z) = P_{i_K}(z)E_{i_K}^z$$

are linearly independent over the complex number field so that the Wronski determinant

$$W(z) = \begin{vmatrix} g_1(z) & g_2(z) & \ldots & g_K(z) \\ g_1'(z) & g_2'(z) & \ldots & g_K'(z) \\ g_1''(z) & g_2''(z) & \ldots & g_K''(z) \\ \cdots & \cdots & \cdots & \cdots \\ g_1^{(K-1)}(z) & g_2^{(K-1)}(z) & \ldots & g_K^{(K-1)}(z) \end{vmatrix}$$

does not vanish identically.

Let now D be the differential operator

$$D = \frac{d}{dz}.$$

By the definition of E_i and by a well known symbolic relation,

$$g_l^{(j)}(z) = \left(\frac{d}{dz}\right)^j \left(P_{i_l}(z)E_{i_l}^z\right) = E_{i_l}^z\left(D + (a_{i_l}/a)\right)^j P_{i_l}(z).$$

Put therefore

$$(7)\qquad P_{ij}(z) = \left(D + (a_i/a)\right)^j P_i(z) \qquad (i = 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots).$$

so that

$$g_l^{(j)}(z) = P_{i_l,j}(z)E_{i_l}^z \qquad (l = 1, 2, \ldots, K;\ j = 0, 1, 2, \ldots).$$

It follows that

$$W(z) = (E_{i_1} E_{i_2} \dots E_{i_K})^z \cdot w(z)$$

where $w(z)$ denotes the new determinant

$$w(z) = \begin{vmatrix} P_{i_1,0}(z) & P_{i_2,0}(z) & \dots & P_{i_K,0}(z) \\ P_{i_1,1}(z) & P_{i_2,1}(z) & \dots & P_{i_K,1}(z) \\ \cdots & \cdots & \cdots & \cdots \\ P_{i_1,K-1}(z) & P_{i_2,K-1}(z) & \dots & P_{i_K,K-1}(z) \end{vmatrix}$$

which naturally also is *not identically zero.*

In this determinant $w(z)$ multiply, for $l = 1, 2, \dots, K$, the lth column by the factor $E_{i_l}^z$, and afterwards add the 2nd, 3rd, ..., Kth new columns to the first new column. This leads to the formula

$$w(z) E_{i_1}^z = \begin{vmatrix} F(z) & P_{i_2,0}(z) & \dots & P_{i_K,0}(z) \\ F'(z) & P_{i_2,1}(z) & \dots & P_{i_K,1}(z) \\ \cdots & \cdots & \cdots & \cdots \\ F^{(K-1)}(z) & P_{i_2,K-1}(z) & \dots & P_{i_K,K-1}(z) \end{vmatrix}$$

because

$$F^{(j)}(z) = \sum_{l=1}^{K} P_{i_l,j}(z) E_{i_l}^z \qquad (j = 0, 1, 2, \dots).$$

On multiplying in this determinant the successive rows by the factors 1, $z, z^2, \dots, z^{K-1}$, respectively, we finally arrive at the equation

$$(8) \quad z^{K(K-1)/2} w(z) E_{i_1}^z = \begin{vmatrix} F(z) & P_{i_2,0}(0) & \dots & P_{i_K,0}(z) \\ zF'(z) & zP_{i_2,1}(z) & \dots & zP_{i_K,1}(z) \\ z^2 F''(z) & z^2 P_{i_2,2}(z) & \dots & z^2 P_{i_K,2}(z) \\ \cdots & \cdots & \cdots & \cdots \\ z^{K-1} F^{(K-1)}(z) & z^{K-1} P_{i_2,K-1}(z) & \dots & z^{K-1} P_{i_K,K-1}(z) \end{vmatrix}.$$

6. By (7), all the $P_{ij}(z)$ are polynomials in z at most of degree r, and hence $w(z)$ is a polynomial in z at most of degree Kr. On the other hand, in the determinant (8), all elements of the first column have at $z = 0$ a zero at least of order m, while, for $l = 2, 3, \dots, K$, all elements of the lth column have at $z = 0$ a zero at least of order $r - r_{i_l}$, respectively. Hence $w(z)$ itself has at $z = 0$ a zero of order not less than

$$\omega = m + \sum_{l=2}^{K} (r - r_{i_l}) - \frac{K(K-1)}{2}.$$

Since $w(z) \neq 0$ is at most of degree Kr, it follows that we can write

$$w(z) = z^{\omega} \Pi(z)$$

where $\Pi(z) \neq 0$ is a polynomial in z at most of degree $s = Kr - \omega$. Naturally, s cannot be negative.

Let us for the moment, without loss of generality, assume that $r = r_1$; this assumption can always be satisfied by a suitable renumbering of the pairs of integers $(a_1, r_1), \ldots, (a_k, r_k)$. *Let us further from now on always assume that*

$$\text{(A)} \qquad r_0 \geqslant R + k^* - k + 1.$$

The first assumption insures that, in explicit form,

$$\text{(9)} \qquad s = Kr - \left(r + \sum_{i=2}^{k} r_i + k - R\right) - \sum_{l=2}^{K} (r - r_{i_l}) + \frac{K(K-1)}{2}$$
$$= \sum_{l=2}^{K} r_{i_l} - \sum_{i=2}^{k} r_i - k + R + \frac{K(K-1)}{2}.$$

The hypothesis (A) *implies that*

$$K = k.$$

For if $K \leqslant k-1$, then there exists a suffix I in the interval $2 \leqslant I \leqslant k$ such that

$$\sum_{l=2}^{K} r_{i_l} - \sum_{i=2}^{k} r_i \leqslant -r_I \leqslant -R - k^* + k - 1,$$

and hence it follows from (9) that $s \leqslant -1$ which is absurd.

Since then $K = k$, and since by our notation we may take $i_1 = 1$, we obtain

$$\sum_{l=2}^{K} r_{i_l} = \sum_{i=2}^{k} r_i,$$

so that the relation (9) leads to the following result.

LEMMA 2. *Assume that the condition* (A) *is satisfied. Then none of the polynomials*

$$P_1(z), \ldots, P_k(z), w(z), \Pi(z)$$

vanishes identically. Here $w(z)$ *is the determinant*

$$w(z) = \begin{vmatrix} P_{10}(z) & P_{20}(z) & \ldots & P_{k0}(z) \\ P_{11}(z) & P_{21}(z) & \ldots & P_{k1}(z) \\ \ldots & \ldots & \ldots & \ldots \\ P_{1,k-1}(z) & P_{2,k-1}(z) & \ldots & P_{k,k-1}(z) \end{vmatrix},$$

and

$$w(z) = z^{\omega}\Pi(z)$$

where $\Pi(z)$ is a polynomial at most of degree

$$s = R + k^* - k.$$

7. The polynomials $P_{ij}(z)$ have been defined by the equations (7). These equations show that they have rational coefficients, hence that the values $P_{ij}(1)$ are rational numbers. In terms of these polynomials, the derivatives

$$F^{(j)}(z) = \sum_{i=1}^{k} P_{ij}(z) E_i^z \qquad (j = 0, 1, 2, \ldots)$$

are linear forms in the k exponential functions $E_1^z, \ldots, E_k^z$.

By Lemma 2, the determinant $w(z)$ of the first k of these linear forms is not identically zero and has at $z = 1$ a zero at most of order $s = R + k^* - k$. Let it in fact have a zero of the exact order σ so that

(10) $w(1) = w'(1) = \ldots = w^{(\sigma-1)}(1) = 0,\; w^{(\sigma)}(1) \neq 0,$ where $0 \leqslant \sigma \leqslant s$.

On solving the first k linear forms

$$F^{(j)}(z) = \sum_{i=1}^{k} P_{ij}(z) E_i^z \qquad (j = 0, 1, \ldots, k-1)$$

for E_i^z, we obtain equations of the form

$$w(z) E_i^z = \sum_{j=0}^{k-1} q_{ij}(z) F^{(j)}(z) \qquad (i = 1, 2, \ldots, k)$$

where the $q_{ij}(z)$ are cofactors of the determinant $w(z)$ and hence are again polynomials in z with rational coefficients.

Differentiate these k equations σ times. Then

$$\sum_{h=0}^{\sigma} \binom{\sigma}{h} w^{(h)}(z)(a_i/a)^{\sigma-h} E_i^z = \sum_{j=0}^{k+\sigma-1} Q_{ij}(z) F^{(j)}(z) \qquad (i = 1, 2, \ldots, k)$$

where also the $Q_{ij}(z)$ are polynomials in z with rational coefficients.

Here finally put $z = 1$. Then, by (10),

$$w^{(\sigma)}(1) E_i = \sum_{j=0}^{k+\sigma-1} Q_{ij}(1) F^{(j)}(1) \qquad (i = 1, 2, \ldots, k).$$

The $k + \sigma$ expressions

$$F^{(j)}(1) = \sum_{i=1}^{k} P_{ij}(1) E_i \qquad (j = 0, 1, \ldots, k+\sigma-1)$$

on the right-hand sides of these equations are linear forms in $E_1, \ldots, E_k$ with rational coefficients, and these $k+\sigma$ linear forms can, by $w^{(\sigma)}(1) \neq 0$, be solved for each of the E_i.

It follows that there exist k distinct suffices $J = J(1),\ J(2), \ldots, J(k)$ in the interval $0 \leqslant J \leqslant k+s-1 = R+k^*-1$ for which the corresponding linear forms

$$F^{(J(j))}(1) = \sum_{i=1}^{k} P_{i,J(j)}(1) E_i \qquad (j = 1, 2, \ldots, k) \tag{11}$$

in $E_1, \ldots, E_k$ are linearly independent. Hence *the determinant of these forms*

$$\Omega = \begin{vmatrix} P_{1,J(1)}(1) & \ldots & P_{k,J(1)}(1) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ P_{1,J(k)}(1) & \ldots & P_{k,J(k)}(1) \end{vmatrix}$$

is distinct from zero.

8. The new polynomials

$$a^j P_{ij}(z) = (aD + a_i)^j P_i(z) \qquad (i = 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots)$$

are again at most of degree r, but have integral rather than rational coefficients, say

$$a^j P_{ij}(z) = \sum_{h=0}^{r} p[h, i, j] z^h \qquad (i = 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots).$$

From

$$P_i(z) = \sum_{h=0}^{r} p(i, j) z^h$$

it follows that $a^j P_{ij}(z)$ has the explicit form

$$a^j P_{ij}(z) = \sum_{h=0}^{r} \sum_{l=0}^{j} \binom{j}{l} a^l a_i^{j-l} p(i, h) h(h-1) \ldots (h-l+1) z^{h-l}$$

$$(i = 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots).$$

Here

$$\sum_{l=0}^{j} \binom{j}{l} a^l |a_i|^{j-l} \leqslant B^j \quad \text{and} \quad h(h-1) \ldots (h-l+1) \leqslant h^l \leqslant r^j,$$

so that

$$\sum_{h=0}^{r} |p[h, i, j]| \leqslant (rB)^j \sum_{h=0}^{r} |p(i, h)|,$$

and therefore, by (5),

$$\sum_{h=0}^{r} |p[h, i, j]| \leqslant (rB)^j 2^r r_i! (k^m B^{m^*})^{R^*} \tag{12}$$

$$(i = 1, 2, \ldots, k; j = 0, 1, 2, \ldots).$$

Put now

$$g_{ij} = a^{J(j)} P_{i,J(j)}(1) \quad (i, j = 1, 2, \ldots, k).$$

Then *all the numbers* g_{ij} *are integers, and by* § 7 *their determinant*

$$g = \begin{vmatrix} g_{11} & \cdots & g_{k1} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ g_{1k} & \cdots & g_{kk} \end{vmatrix} = a^{k^*} \Omega$$

does not vanish.

Since none of the suffices $J(j)$ exceeds $R + k^* - 1$, we deduce immediately from the estimate (12) that

$$|g_{ij}| \leqslant C_1 r_i! \quad (i, j = 1, 2, \ldots, k) \tag{13}$$

where C_1 denotes the expression

$$C_1 = 2^r (rB)^{R+k^*-1} (k^m B^{m^*})^{R^*}. \tag{14}$$

9. In analogy to the integers g_{ij} put

$$L_j = a^{J(j)} F^{(J(j))}(1) \quad (j = 1, 2, \ldots, k),$$

so that L_j is the linear form

$$L_j = g_{1j} E_1 + \ldots + g_{kj} E_k \quad (j = 1, 2, \ldots, k)$$

in $E_1, \ldots, E_k$. An upper estimate for $|L_j|$ is obtained as follows.

The hypothesis

$$r_0 \geqslant R + k^* - k + 1 \tag{A}$$

implies that

$$m = r_1 + \ldots + r_k + k - R \geqslant k(R + k^* - k + 1) + k - R$$
$$= (k-1) R + (k-2) k^* + k,$$

hence, by $k \geqslant 2$, that

$$m > R + k^* - 1. \tag{15}$$

From

$$F(z) = r! \sum_{h=m}^{\infty} f_h \frac{z^h}{h!}$$

it follows further that

$$F^{(j)}(z) = r! \sum_{h=m}^{\infty} f_h \frac{z^{h-j}}{(h-j)!}.$$

Here we proved already the estimate

$$(6) \qquad |f_h| \leqslant k(B/a)^h (k^m B^{m^*})^{R^*} \quad \text{for} \quad h \geqslant m.$$

Hence it follows that

$$|a^j F^{(j)}(1)| \leqslant a^j r! \sum_{h=m}^{\infty} k(B/a)^h (k^m B^{m^*})^{R^*} \frac{1}{(h-j)!}.$$

Here substitute $h = m + l$ in the infinite series; the right-hand side assumes then the form

$$\frac{a^j kr!}{(m-j)!} (B/a)^m (k^m B^{m^*})^{R^*} \sum_{l=0}^{\infty} \frac{(B/a)^l}{(m-j+1)(m-j+2)\ldots(m-j+l)}$$

where for $j \leqslant m$ the infinite series satisfies the inequality

$$\sum_{l=0}^{\infty} \leqslant e^{B/a}.$$

Finally let j run over the suffices $J(1), \ldots, J(k)$. These suffices do not exceed $R + k^* - 1$, hence by (15) are less than m. Thus we obtain the estimate

$$(16) \qquad |L_j| \leqslant \frac{a^{R+k^*-1} e^{B/a} kr!}{(m-R-k^*-1)!} (B/a)^m (k^m B^{m^*})^{R^*} \qquad (j = 1, 2, \ldots, k).$$

Assume now again, just as in § 6, that r_1 is the largest of the integers $r_1, \ldots, r_k$, thus that $r = r_1$. By (16),

$$|L_j r_2! \ldots r_k!| \leqslant \frac{r_2! \ldots r_k!}{(m-R-k^*-1)!} ke^{B/a} a^{R+k^*-1-m} (k^m B^{m^*})^{R^*}.$$

Here, by (15),

$$a^{R+k^*-1-m} \leqslant 1.$$

Further

$$0 < m - R - k^* - 1 = (r_1 + \ldots + r_k) - (2R + k^* - k - 1)$$

$$\text{and} \qquad r_1 + \ldots + r_k \leqslant kr,$$

hence

$$(m - R - k^* - 1)! \geqslant (r_1 + \ldots + r_k)!(kr)^{-(2R+k^*-k-1)}.$$

Also

$$\frac{r_1! \dots r_k!}{(r_1 + \dots + r_k)!} \leqslant 1$$

because the reciprocal of this fraction is an integer. It follows then that

(17) $$|L_j r_2! \dots r_k!| \leqslant C_2 \qquad (j = 1, 2, \dots, k)$$

where C_2 denotes the expression

(18) $$C_2 = ke^{B/a}(kr)^{2R+k^*-k-1} B^m (k^m B^{m^*})^{R^*}.$$

The results so proved in this and the preceding section may be combined into the following lemma.

LEMMA 3. *Let the notation be as in* § 2 *and assume in addition that*

$$r = r_1 \qquad \text{and} \qquad r_0 \geqslant R + k^* - k + 1.$$

Then there exist k linearly independent linear forms

$$L_j = g_{1j} E_1 + \dots + g_{kj} E_k \qquad (j = 1, 2, \dots, k)$$

with integral coefficients g_{ij} *such that*

$$|g_{ij}| \leqslant C_1 r_i! \qquad (i, j = 1, 2, \dots, k),$$

$$|L_j r_2! \dots r_k!| \leqslant C_2 \qquad (j = 1, 2, \dots, k),$$

where C_1 *and* C_2 *are defined by* (14) *and* (18), *respectively.*

10. Lemma 3 will now be applied to the study of a general linear form. Denote by

$$L = x_1 E_1 + \dots + x_k E_k$$

a linear form in $E_1, \dots, E_k$ with integral coefficients not all zero, and put

$$x_j' = 1 \text{ if } x_j = 0, \qquad \text{and} \qquad x_j' = x_j \text{ if } x_j \neq 0 \qquad (j = 1, 2, \dots, k),$$

and

$$x = \max(|x_1|, \dots, |x_k|) = \max(|x_1'|, \dots, |x_k'|).$$

We shall now choose the parameters $r_1, \dots, r_k$, R of Lemma 3 as functions of $x_1', \dots, x_k'$ by the following construction.

Put

$$C = C(r) = k^2 r \big((\log B)(\log r)\big)^{1/2},$$

and define a function $f(r)$ of the positive integer r by

$$f(r) = e^{-2C(r)} r!.$$

A well known form of Sterling's formula states that

$$r! = \sqrt{2\pi r}\, r^r e^{-r+\varrho(r)}, \quad \text{where} \quad 0 < \varrho(r) < \frac{1}{12r}.$$

It follows that

$$\frac{\log f(r)}{r} = \log r - 2k^2\big((\log B)(\log r)\big)^{1/2} - 1 + \sigma(r),$$

where $\sigma(r)$ denotes the expression

$$\sigma(r) = \frac{\log r}{2r} + \frac{\log 2\pi}{2r} + \frac{\varrho(r)}{r}.$$

Here, for $r \geqslant 2$, it is easily verified that

$$0 < \sigma(r) < 1,$$

hence that

(19)

$$\log r - 2k^2\big((\log B)(\log r)\big)^{1/2} - 1 < \frac{\log f(r)}{r} < \log r - 2k^2\big((\log B)(\log r)\big)^{1/2}.$$

The definition of $f(r)$ and this inequality show immediately that

(20) $$f(1) = 1; \quad f(r) < 1 \quad \text{if} \quad 2 \leqslant r \leqslant B^{4k^4}.$$

It is also obvious that

(21) $$C(r-1) < C(r) \quad \text{if} \quad r \geqslant 2.$$

By definition, x is a positive integer. There exists therefore a *smallest* positive integer r such that

$$f(r) > x,$$

and this integer necessarily has the further properties

(22) $$f(r-1) \leqslant x < f(r),$$

so that by (21) also

(23) $$(r-1)! \leqslant e^{2C(r)}x < r!.$$

Define similarly the integers $r_1, \ldots, r_k$ by the inequalities

(24) $$(r_j - 1)! \leqslant e^{2C(r)}|x_j'| < r_j! \quad (j = 1, 2, \ldots, k).$$

Then by (23) and (24) and in agreement with the hypothesis of § 2,

$$r = \max(r_1, \ldots, r_k).$$

Without loss of generality, let from now on

$$x = |x_1'| = |x_1|$$

be the largest of the integers $|x_1'|, \ldots, |x_k'|$. The formulae (23) and (24) imply then that also

$$r = r_1$$

is the largest of the integers $r_1, \ldots, r_k$, in agreement with the previous assumption.

By (22), $f(r)$ is greater than $x \geqslant 1$. Hence, by (20) necessarily

$$r \geqslant B^{4k^4} + 1. \tag{25}$$

11. Having fixed $r, r_1, \ldots, r_k$ in this manner, define now R by

$$R = \left[kr \left(\frac{\log B}{\log r} \right)^{1/2} \right] + 1, \tag{26}$$

so that

$$kr \left(\frac{\log B}{\log r} \right)^{1/2} < R \leqslant kr \left(\frac{\log B}{\log r} \right)^{1/2} + 1.$$

By (25) and $k \geqslant 2$ this choice implies that

$$R < \frac{k}{2k^2} r + 1 < r + k - 1,$$

and since $r(\log r)^{-1/2}$ is an increasing function of r when (25) holds, it also follows from $B \geqslant 2$ that

$$R > \frac{B^{4k^4}}{2k} \geqslant \frac{2^{4k^4}}{2k} > \frac{4k^4}{2k} > \max(k, k^*). \tag{27}$$

Hence the condition

$$k \leqslant R \leqslant r_1 + \ldots + r_k + k - 1$$

of § 2 is certainly satisfied. It further follows that

$$R + k^* - k + 1 < 2R.$$

The former hypothesis

$$r_0 \geqslant R + k^* - k + 1 \tag{A}$$

does then certainly hold if

$$r_j \geqslant 2R \qquad (j = 1, 2, \ldots, k).$$

That this set of inequalities is in fact satisfied will now be proved indirectly. Assume there exists a suffix j for which

$$r_j < 2R.$$

Then, by (26) and (27),

$$r_j < 2R \leqslant 2kr\left(\frac{\log B}{\log r}\right)^{1/2} + 2 < 3kr\left(\frac{\log B}{\log r}\right)^{1/2},$$

whence

$$\log r_j! \leqslant r_j \log r_j < 3kr\left(\frac{\log B}{\log r}\right)^{1/2} \log\left(3kr\left(\frac{\log B}{\log r}\right)^{1/2}\right).$$

Here, again by (25) and by $k \geqslant 2$,

$$3k\left(\frac{\log B}{\log r}\right)^{1/2} < \frac{3k}{2k^2} < 1.$$

Hence

$$\log r_j! < 3kr\left(\frac{\log B}{\log r}\right)^{1/2} \cdot \log r = 3kr\left((\log B)(\log r)\right)^{1/2},$$

and so, once more by $k \geqslant 2$,

$$r_j! < e^{2C(r)},$$

contrary to the definition (24) of r_j because $|x_j'|$ is at least 1.

We have thus proved that the definitions (22), (24), and (26) of $r, r_1, \ldots, r_k$, and R, together with a notation such that $x = |x_1'|$ and hence also $r = r_1$, satisfy all the conditions of § 2 and of Lemma 3. We are then allowed to apply this lemma.

12. This means that, in addition to the given linear form

$$L = x_1 E_1 + \ldots + x_k E_k,$$

there exist the k linearly independent linear forms

$$L_j = g_{1j} E_1 + \ldots + g_{kj} E_k \quad (j = 1, 2, \ldots, k)$$

of the lemma which have integral coefficients such that

$$(28) \qquad \begin{aligned} &|g_{ij}| \leqslant C_1 r_i! \quad (i, j = 1, 2, \ldots, k), \\ &|L_j r_2! \ldots r_k!| \leqslant C_2 \quad (j = 1, 2, \ldots, k). \end{aligned}$$

The form L is then linearly independent of certain $k-1$ of the forms L_j. To fix the ideas, assume that the k forms

$$(29) \qquad L, L_2, \ldots, L_k$$

are linearly independent. Hence their determinant

$$\begin{vmatrix} x_1 & \ldots & x_k \\ g_{12} & \ldots & g_{k2} \\ \cdot & \cdot & \cdot \\ g_{1k} & \ldots & g_{kk} \end{vmatrix}, = \varDelta \text{ say},$$

does not vanish. This determinant is an integer, and so

$$|\Delta| \geqslant 1.$$

On solving the k forms (29), say for E_1, we find that

$$\begin{vmatrix} L & x_2 & \dots & x_k \\ L_2 & g_{22} & \dots & g_{k2} \\ \cdot & \cdot & \cdot & \cdot \\ L_k & g_{2k} & \dots & g_{kk} \end{vmatrix} = \Delta E_1.$$

It follows that

$$(30) \qquad \Delta e^{a_1/a} = LM + L_2 M_2 + \dots + L_k M_k,$$

where $M, M_2, \dots, M_k$ denote the cofactors of the elements $L, L_2, \dots, L_k$ in the first column of the determinant for ΔE_1, respectively.

By (28),

$$|M| \leqslant (k-1)!\, C_1^{k-1} r_2! \dots r_k!$$

and since $|x_j| \leqslant |x_j'|$,

$$|M_j| \leqslant (k-2)!\, C_1^{k-2} C_2 r_2! \dots r_k! \sum_{l=2}^{k} \frac{|x_l'|}{r_l!} \qquad (j = 2, 3, \dots, k).$$

The identity (30) implies therefore the inequality

$$(31) \qquad 1 \leqslant U + V,$$

where

$$(32) \quad U = (k-1)!\, e^{-a_1/a} C_1^{k-1} r_2! \dots r_k! \, |L|, \qquad V = (k-1)!\, e^{-a_1/a} C_1^{k-2} C_2 \sum_{l=2}^{k} \frac{|x_l'|}{r_l!}.$$

We shall next establish upper estimates for U and V.

13. Since $r_j \leqslant r$ for all j, by (24),

$$r_2! \dots r_k! \leqslant r^{k-1} e^{2(k-1)C(r)} |x_2' \dots x_k'|, \qquad \sum_{l=2}^{k} \frac{|x_l'|}{r_l!} \leqslant (k-1)\, e^{-2C(r)},$$

and therefore

$$U \leqslant (k-1)!\, e^{-a_1/a} C_1^{k-1} r^{k-1} e^{2(k-1)C(r)} |x_2' \dots x_k' L|, \qquad V \leqslant k!\, e^{-a_1/a} C_1^{k-2} C_2 e^{-2C(r)}.$$

Here, by the definitions (14) and (18),

$$C_1 = 2^r (rB)^{R+k^*-1} (k^m B^{m^*})^{R^*}, \qquad C_2 = k e^{B/a} (kr)^{2R+k^*-k-1} B^m (k^m B^{m^*})^{R^*}.$$

It follows that

$$(33) \qquad U \leqslant C_3 |x_2' \ldots x_k' L|, \qquad V \leqslant C_4$$

where C_3 and C_4 are defined by

$$C_3 = (k-1)!\, e^{-a_1/a}\big(2^r (rB)^{R+k^*-1} (k^m B^{m^*})^{R^*}\big)^{k-1} r^{k-1} e^{2(k-1)C(r)}$$

and

$$C_4 = k!\, e^{-a_1/a}\big(2^r (rB)^{R+k^*-1} (k^m B^{m^*})^{R^*}\big)^{k-2} k e^{B/a} (kr)^{2R+k^*-k-1} B^m (k^m B^{m^*})^{R^*} e^{-2C(r)}.$$

Here it is convenient to split off the factors of maximal size from C_3 and C_4 and to write these expressions as

$$(34) \qquad C_3 = C_5 e^{2(k-1)C(r)} r^{(k-1)(R-1)} B^{(k-1)m^*R^*}, \qquad C_4 = C_6 e^{-2C(r)} r^{k(R-1)} B^{(k-1)m^*R^*},$$

where the new factors C_5 and C_6 are given by

$$C_5 = (k-1)!\, e^{-a_1/a} 2^{(k-1)r} k^{(k-1)mR^*} r^{(k-1)(k^*+1)} B^{(k-1)(R+k^*-1)}$$

and

$$C_6 = k \cdot k!\, e^{(B-a_1)/a} 2^{(k-2)r} k^{2R+(k-1)mR^*+k^*-k-1} r^{(k-1)k^*-k+1} B^{(k-2)(R+k^*-1)+m}.$$

The next step consists in obtaining simple upper estimates for $2C_5$ and $2C_6$ and hence also for $2C_3$ and $2C_4$.

14. Firstly, by the definition (26) of R,

$$R-1 \leqslant kr\left(\frac{\log B}{\log r}\right)^{1/2} \leqslant R,$$

while

$$m \leqslant kr \quad \text{and therefore} \quad m^* \leqslant \frac{k^2 r^2}{2}.$$

The second factors of C_3 and C_4 in (34) have therefore the upper bounds

$$(35) \qquad e^{2(k-1)C(r)} r^{(k-1)(R-1)} B^{(k-1)m^*R^*} \leqslant e^{\left(2k-\frac{1}{2}-\frac{3}{2k}\right)C(r)}$$

and

$$(36) \qquad e^{-2C(r)} r^{k(R-1)} B^{(k-1)m^*R^*} \leqslant e^{-\left(\frac{1}{2}+\frac{1}{2k}\right)C(r)}.$$

To deal with the first factors C_5 and C_6, we first note that

$$2(k-1)! \leqslant k^k, \qquad k\cdot k! \leqslant k^k, \qquad (k-1)(k^*+1) \leqslant k^3/2,$$
$$(k-1)k^* - k + 1 \leqslant k^3/2, \qquad R + k^* - 1 \leqslant 2R$$

and

$$e^{-a_1/a} \leqslant e^B, \qquad e^{(B-a_1)/a} \leqslant e^{2B}.$$

Therefore

$$2C_5 \leqslant k^k e^B 2^{kr} k^{k^2 r R^*} r^{k^3/2} B^{2kR}, \quad = e^{C_7} \text{ say},$$

and

$$2C_6 \leqslant k^k e^{2B} 2^{kr} k^{2R+k^2 r R^* + k^2/2} r^{k^3/2} B^{2kR+kr}, \quad = e^{C_8} \text{ say}.$$

Here

$$k \geqslant 2, \quad \log k \leqslant e^{-1} k < k, \quad B \geqslant 2, \quad \log B \leqslant e^{-1} B < B, \quad 2 \cdot \log B > 1.$$

Also, by (25),

$$r > B^{4k^4} \geqslant 2^{64} > e^{32}, \quad \log r > 4k^4 \log B > 2k^4 \geqslant 32.$$

Therefore

$$\big((\log B)(\log r)\big)^{1/2} \geqslant 2k^2 \log B > k^2,$$

whence

$$C(r) = k^2 r \big((\log B)(\log r)\big)^{1/2} > k^4 r > k^4 B^{4k^4} \geqslant 2^{68}.$$

We also note that, for the values of r considered, the function

$$\frac{\log r}{r}$$

is strictly decreasing and hence satisfies the inequality

$$\frac{\log r}{r} < \frac{4k^4 \log B}{B^{4k^4}} < \frac{4k^4}{B^{4k^4-1}} \leqslant \frac{2^6}{2^{63}} = 2^{-57}.$$

Thus the following upper estimates for the successive terms of C_7 and C_8 are obtained.

$$C(r)^{-1} \cdot k \cdot \log k < \frac{k^2}{k^4 r} = \frac{1}{k^2 r} < \frac{1}{2^2 \cdot 2^{64}} = 2^{-68};$$

$$C(r)^{-1} \cdot 2B < \frac{2B}{k^4 B^{4k^4}} \leqslant \frac{1}{k^3 B^{4k^4-1}} \leqslant \frac{1}{2^3 \cdot 2^{63}} = 2^{-66};$$

$$C(r)^{-1} \cdot kr \cdot \log 2 < \frac{\log 2}{k\big((\log B)(\log r)\big)^{1/2}} \leqslant \frac{\log 2}{k \cdot 2k^2 \log B} \leqslant \frac{1}{2k^3} \leqslant \frac{1}{16};$$

$$C(r)^{-1} \cdot 2R \cdot \log k < \frac{3kr\left(\dfrac{\log B}{\log r}\right)^{1/2} \cdot e^{-1} k}{k^2 r \big((\log B)(\log r)\big)^{1/2}} = \frac{3}{ek \cdot \log r} < \frac{3}{64e} < \frac{1}{40};$$

$$C(r)^{-1}\cdot k^2rR^*\log k < \frac{k^2r\cdot\log k}{kr\left(\dfrac{\log B}{\log r}\right)^{1/2}\cdot k^2r\left((\log B)(\log r)\right)^{1/2}}$$

$$= \frac{\log k}{kr\cdot\log B} \leqslant \frac{1}{er\cdot\log 2} < \frac{1}{r} < 2^{-64};$$

$$C(r)^{-1}\cdot\frac{k^2}{2}\log k < \frac{k^2\cdot k}{2k^4r} = \frac{1}{2kr} < 2^{-66};$$

$$C(r)^{-1}\cdot\frac{k^3}{2}\log r < \frac{k^3\log r}{2k^4r} = \frac{\log r}{2kr} \leqslant \frac{\log r}{4r} < 2^{-59};$$

$$C(r)^{-1}\cdot 2kr\cdot\log B < \frac{k\cdot 3kr\left(\dfrac{\log B}{\log r}\right)^{1/2}\cdot\log B}{k^2r\left((\log B)(\log r)\right)^{1/2}} = \frac{3\cdot\log B}{\log r}$$

$$< \frac{3\cdot\log B}{4k^4\cdot\log B} = \frac{3}{4k^4} < \frac{1}{16};$$

$$C(r)^{-1}\cdot kr\cdot\log B = \frac{kr\cdot\log B}{k^2\left((\log B)(\log r)\right)^{1/2}} = \frac{1}{k}\left(\frac{\log B}{\log r}\right)^{1/2} < \frac{1}{k\cdot 2k^2} \leqslant \frac{1}{16}.$$

On adding these results it follows at once that

$$C_7 < \tfrac{1}{4}C(r) \quad\text{and}\quad C_8 < \tfrac{1}{4}C(r),$$

hence that

$$2C_5 < e^{\frac{1}{4}C(r)} \quad\text{and}\quad 2C_6 < e^{\frac{1}{4}C(r)}.$$

Therefore, by (34), (35), and (36),

(37) $$2C_3 < e^{\left(2k-\frac{1}{4}-\frac{3}{2k}\right)C(r)} < e^{\left(2k-\frac{1}{4}\right)C(r)} < e^{2kC(r)}$$

and

(38) $$2C_4 < e^{-\left(\frac{1}{4}+\frac{1}{2k}\right)C(r)} < e^{-\frac{1}{4}C(r)} < 1.$$

By (33), these estimates imply that

$$2U < e^{2kC(r)}|x_2' \ldots x_k' L| \quad\text{and}\quad 2V < 1.$$

(In fact, they imply the slightly stronger inequalities

(39) $$2U < e^{(2k-\frac{1}{4})C(r)}|x_2' \ldots x_k' L| \quad\text{and}\quad 2V < e^{-\frac{1}{4}C(r)}.)$$

Since $2V < 1$, it next follows from (31) that

$$2U > 1.$$

This we combine with the upper bound for $2U$ just obtained, and we multiply both sides of the resulting inequality by the factor $x = |x_1|$. We may then again drop the hypothesis that $|x_1| = \max(|x_1|, \ldots, |x_k|)$ and so obtain the following result.

THEOREM 1. *Let $a_1, \ldots, a_k$, where $k \geqslant 2$, be distinct integers, and let $a > 0$ be an integer satisfying $(a, a_1, \ldots, a_k) = 1$; let further $x_1, \ldots, x_k$ be integers not all zero. Put*

$$B = a + \max(|a_1|, \ldots, |a_k|), \qquad E_1 = e^{a_1/a}, \ldots, E_k = e^{a_k/a}$$

and

$$x_j' = \begin{cases} 1 & \text{if} \quad x_j = 0 \\ x_j & \text{if} \quad x_j \neq 0 \end{cases} \qquad (j = 1, 2, \ldots, k);$$

$$x = \max(|x_1|, \ldots, |x_k|).$$

Also, for positive integral r, put

$$C(r) = k^2 r\big((\log B)(\log r)\big)^{1/2}, \qquad f(r) = e^{-2C(r)} r!.$$

If r denotes the smallest positive integer for which

$$f(r-1) \leqslant x < f(r),$$

then

$$r \geqslant B^{4k^4} + 1$$

and

$$|x_1' x_2' \ldots x_k' (x_1 E_1 + x_2 E_2 + \ldots + x_k E_k)| > x e^{-2kC(r)}.$$

By (39), this inequality may in fact be replaced by

(40) $$|x_1' x_2' \ldots x_k' (x_1 E_1 + x_2 E_2 + \ldots + x_k E_k)| > x e^{-(2k-\frac{1}{2})C(r)},$$

which is slightly stronger.

15. As a corollary to this theorem we show how it simplifies when x is very large, thus under the hypothesis that, say

(41) $$x \geqslant B^{16k^4 \cdot B^{16k^4}}.$$

It had been found in § 10, formula (19), that

$$\log r - 2k^2\big((\log B)(\log r)\big)^{1/2} - 1 < \frac{\log f(r)}{r} < \log r - 2k^2\big((\log B)(\log r)\big)^{1/2}.$$

Here the right-hand side implies that

$$f(r) < r^r.$$

Since $f(r) > x$, it follows therefore from (41) that now

$$r \geqslant B^{16k^4}+1,$$

hence, by $k \geqslant 2$ and $B \geqslant 2$, that

(42) $$r > 2^{256}.$$

By the first lower bound for r,

$$(\log B)^{1/2} \leqslant \frac{(\log(r-1))^{1/2}}{4k^2},$$

whence, by (19),

$$\frac{f(r-1)}{r-1} > \frac{1}{2}\log(r-1)-1.$$

This implies that

$$\frac{f(r-1)}{r-1} > \frac{r-1}{r}\left(\tfrac{1}{2}\log(r-1)-1\right) > \tfrac{1}{3}\log r,$$

as follows from the very large lower bound (42) for r. Thus also

$$f(r-1) > r^{r/3}.$$

The integer r is then connected with x by the inequalities

$$r^{r/3} < x < r^r,$$

so that

$$\frac{r}{3}\log r < \log x < r\cdot\log r,$$

$$\log r - \log 3 + \log\log r < \log\log x < \log r + \log\log r \leqslant 2\cdot\log r.$$

On the left-hand side, by (42), trivially

$$\log\log r > \log 3,$$

so that

$$\log r < \log\log x < 2\cdot\log r.$$

On combining the last inequalities, it follows then that

$$\frac{\log x}{\log\log x} < r < \frac{6\cdot\log x}{\log\log x}.$$

These inequalities combine to the result that

$$C(r) < 6k^2(\log x)\left((\log B)(\log\log x)^{-1}\right)^{1/2}.$$

Theorem 1 implies therefore the following corollary.

Let $a_1, \ldots, a_k, a, B, E_1, \ldots, E_k, x_1, \ldots, x_k, x_1', \ldots, x_k'$, *and* x *be as in Theorem* 1, *but assume that now*

$$x \geqslant B^{16k^4 \cdot B^{16k^4}}.$$

Then

$$|x_1' \ldots x_k'(x_1 E_1 + \ldots + x_k E_k)| > x \cdot e^{-12k^3(\log x)((\log B)(\log\log x)^{-1})^{1/2}}.$$

This is Baker's first result, but with explicit constants.

16. Let $a_1, \ldots, a_k, a, E_1, \ldots, E_k$ be as in Theorem 1, but assume that a_k and E_k have now been specialized by taking

$$a_k = 0, \quad \text{hence} \quad E_k = 1.$$

Denote by $y_1, \ldots, y_k$ positive integers such that

(43) $$y_k \geqslant k$$

and that the product

$$\omega = y_k |y_k E_1 - y_1| \ldots |y_k E_{k-1} - y_{k-1}|$$

satisfies the inequality

(44) $$0 < \omega < 1.$$

Theorem 1 will enable us to establish a lower bound for ω in terms of y_k.

For this purpose put

$$\varphi_j = \omega^{1/(k-1)} |y_k E_j - y_j| \quad (j = 1, 2, \ldots, k-1)$$

and assume, without loss of generality, that the notation is such that

$$\varphi_1 \geqslant \varphi_2 \geqslant \ldots \geqslant \varphi_{k-1} > 0.$$

Since evidently

(45) $$\varphi_1 \varphi_2 \ldots \varphi_{k-1} = y_k \geqslant k \geqslant 2,$$

not all the φ_j can be $\leqslant 1$.

If

$$\varphi_{k-1} > 1 \quad \text{and hence} \quad \varphi_j > 1 \quad \text{for} \quad j = 1, 2, \ldots, k-1,$$

put $\varkappa = k-1$; otherwise denote by $\varkappa$ the smallest suffix in the interval

$$1 \leqslant \varkappa \leqslant k-2$$

for which

$$\varphi_{\varkappa+1} \varphi_{\varkappa+2} \ldots \varphi_{k-1} \leqslant 1.$$

By (45), such a suffix certainly exists.

17. Having fixed $\varkappa$ in this way, consider now the system of $\varkappa+1 \leqslant k$ linear inequalities,

$$|x_j| \leqslant \varphi_j \quad (j = 1, 2, \ldots, \varkappa-1),$$

(46)
$$|x_\varkappa| \leqslant \varphi_\varkappa \varphi_{\varkappa+1} \cdots \varphi_{k-1} \leqslant \varphi_\varkappa,$$

$$|x_1 y_1 + \ldots + x_\varkappa y_\varkappa + x_k y_k| < 1$$

for $x_1, x_2, \ldots, x_\varkappa, x_k$. The $\varkappa+1$ linear forms

$$x_1, x_2, \ldots, x_\varkappa, x_1 y_1 + \ldots + x_\varkappa y_\varkappa + x_k y_k$$

in $x_1, x_2, \ldots, x_\varkappa, x_k$ on the left-hand sides in (46) have the determinant y_k; and the product of the right-hand sides is by (45) equal to the same value since

$$\varphi_1 \cdots \varphi_{\varkappa-1} \cdot \varphi_\varkappa \varphi_{\varkappa+1} \cdots \varphi_{k-1} = y_k.$$

Hence, by Minkowski's theorem on linear forms, the inequalities (46) can be satisfies by a system of $\varkappa+1$ integers $x_1, x_2, \ldots, x_\varkappa, x_k$ not all zero. But since all the x's and y's are integers, the last inequality (46) implies the equation

(47)
$$x_1 y_1 + \ldots + x_\varkappa y_\varkappa + x_k y_k = 0.$$

Hence it follows that already *at least one of the integers*

$$x_1, x_2, \ldots, x_\varkappa$$

does not vanish. On the other hand, it is uncertain, and in fact of no importance, whether x_k is or is not equal to zero.

We denote from now on by $i_1, i_2, \ldots, i_K$ all the distinct suffices

$$1, 2, \ldots, \varkappa$$

for which

$$x_{i_l} \neq 0 \quad (l = 1, 2, \ldots, K);$$

here naturally

$$1 \leqslant K \leqslant \varkappa.$$

18. The right-hand sides φ_j and $\varphi_\varkappa \varphi_{\varkappa+1} \cdots \varphi_{k-1}$ of the first $\varkappa$ inequalities (46) all are greater than 1. It follows therefore from these inequalities and from the equation (45) that

(48)
$$|x_{i_1} x_{i_2} \cdots x_{i_K}| \leqslant \varphi_1 \varphi_2 \cdots \varphi_{\varkappa-1} \cdot \varphi_\varkappa \varphi_{\varkappa+1} \cdots \varphi_{k-1} = y_k.$$

It is also possible to give an upper bound for x_k. For identically,

$$x_1 y_1 + \ldots + x_\varkappa y_\varkappa + x_k y_k = (x_1 E_1 + \ldots + x_\varkappa E_\varkappa + x_k) y_k - \sum_{j=1}^{\varkappa} x_j (y_k E_j - y_j),$$

so that, by (47),

$$(49)\qquad (x_1E_1+\dots+x_\varkappa E_\varkappa+x_k)y_k = \sum_{j=1}^{\varkappa} x_j(y_kE_j-y_j).$$

Here

$$|x_j|\leqslant \varphi_j,\qquad |y_kE_j-y_j| = \frac{\omega^{1/(k-1)}}{\varphi_j}\qquad (j=1,2,\dots,\varkappa).$$

It follows then from (49) that

$$|x_1E_1+\dots+x_\varkappa E_\varkappa+x_k|\leqslant \varkappa\cdot\omega^{1/(k-1)}.$$

Here $\varkappa\leqslant k-1$, $\omega<1$, and $y_k\geqslant k$, so that also

$$(50)\qquad |x_1E_1+\dots+x_\varkappa E_\varkappa+x_k|\leqslant \frac{(k-1)\,\omega^{1/(k-1)}}{y_k}<1.$$

Thus

$$|x_k|<1+|x_1|E_1+\dots+|x_\varkappa|E_\varkappa,$$

where

$$E_j = e^{a_j/a}\leqslant e^{A/a}\leqslant e^B\qquad (j=1,2,\dots,\varkappa).$$

Hence, by $\varkappa\leqslant k-1$,

$$|x_k|<ke^B\max(|x_1|,\dots,|x_\varkappa|) = ke^B\max(|x_{i_1}|,\dots,|x_{i_K}|).$$

Therefore, on noting that $x_{\varkappa+1}=x_{\varkappa+2}=\dots=x_{k-1}=0$ and putting

$$x = \max(|x_1|,\dots,|x_k|) = \max(|x_{i_1}|,\dots,|x_{i_K}|,|x_k|),$$

it has been proved that

$$(51)\qquad x<ke^B\max(|x_{i_1}|,\dots,|x_{i_K}|).$$

All factors of the product $x_{i_1}x_{i_2}\dots x_{i_K}$ are integers not zero so that the absolute value of this product cannot be less than $\max(|x_{i_1}|,\dots,|x_{i_K}|)$. The inequality (51) implies therefore that

$$|x_{i_1}x_{i_2}\dots x_{i_K}|\geqslant (ke^B)^{-1}x,$$

and hence it follows from (48) that

$$(52)\qquad x\leqslant ke^By_k.$$

Put now again

$$x_j' = \begin{cases}1 & \text{if}\quad x_j=0\\ x_j & \text{if}\quad x_j\neq 0\end{cases}\qquad (j=1,2,\dots,k).$$

Since $|x_k| \leqslant x$, by (48),

$$|x_1' x_2' \ldots x_k'| \leqslant x y_k,$$

whence, by (50),

(53) $$|x_1' x_2' \ldots x_k'(x_1 E_1 + \ldots + x_{k-1} E_{k-1} + x_k| \leqslant (k-1) x \omega^{1/(k-1)}.$$

19. Apply now to this inequality (53) the remark to Theorem 1. For this purpose, with a slight change of notation, denote by r' and r'' the smallest positive integers satisfying

$$f(r'-1) \leqslant x < f(r') \quad \text{and} \quad f(r''-1) \leqslant y_k < f(r''),$$

respectively. It follows immediately from the estimate (40) of § 14 that

$$(k-1)\,\omega^{1/(k-1)} > e^{-\left(2k-\frac{1}{4}\right)C(r')},$$

so that, by the definition of ω,

(54) $$y_k |y_k E_1 - y_1| \ldots |y_k E_{k-1} - y_{k-1}| > (k-1)^{-(k-1)} e^{-(k-1)\left(2k-\frac{1}{4}\right)C(r')}.$$

This formula has still the disadvantage of involving the integer r' depending on x rather than the integer r'' which depends on y_k. We show now how to change over to a formula involving r''.

For the moment, put

$$\beta = 2k^2 (\log B)^{1/2}.$$

In § 10 we had for every integer $r \geqslant 2$ obtained the formula

$$\frac{\log f(r)}{r} = \log r - \beta (\log r)^{1/2} - 1 + \sigma(r),$$

where

$$\sigma(r) = \frac{\log(2\pi r)}{2r} + \frac{\varrho(r)}{r} \quad \text{and} \quad 0 < \varrho(r) < \frac{1}{12r}.$$

Assume now again that

$$r \geqslant B^{4k^4} + 1;$$

then, by § 14,

$$r > 2^{64}, \quad \log r > \beta^2 > 32, \quad \frac{\log r}{r} < 2^{-57}.$$

From these estimates it is easily deduced that

$$0 < \sigma(r) < \frac{\log r}{r} < 2^{-57},$$

and that therefore

$$\frac{\log f(r)}{r} < \log r - \beta(\log r)^{1/2} - 1 + 2^{-57}.$$

Hence, whenever $f(r) > 1$, then necessarily

(55) $$\log r - \beta(\log r)^{1/2} > 1 - 2^{-57} > \tfrac{3}{4}.$$

On the other hand, from the definition of $f(r)$,

$$\frac{f(r+1)}{f(r)} = e^{\log(r+1) - \beta(\log(r+1))^{1/2}} \cdot e^{-\beta r((\log[r+1])^{1/2} - (\log r)^{1/2})}.$$

Here, by (55) applied to $r+1$ instead of r, the first exponential factor on the right-hand side is greater than $e^{3/4}$. Next, by the mean value theorem of differential calculus,

$$\left(\log(r+1)\right)^{1/2} - (\log r)^{1/2} < \left(2r(\log r)^{1/2}\right)^{-1},$$

hence, by $\log r > \beta^2$,

$$\beta r\left((\log[r+1])^{1/2} - (\log r)^{1/2}\right) < \frac{\beta}{2\beta} = \frac{1}{2}$$

so that the second exponential factor is greater than $e^{-1/2}$. We have thus found the basic inequality

(56) $$f(r+1) > e^{1/4} f(r) \quad \text{if} \quad r \geqslant B^{4k^4} + 1.$$

20. This inequality shows that $f(r)$ is strictly increasing and that for every pair of positive integers n and $r \geqslant B^{4k^4} + 1$,

$$f(r+n) > e^{n/4} f(r).$$

Now, by (52) and by the definitions of r' and r'',

$$f(r'-1) \leqslant k e^B y_k \quad \text{and} \quad y_k > f(r'').$$

It follows that

$$r' < r'' + 4B + 4 \cdot \log k + 1.$$

The right-hand side of the estimate (54) is then greater than

$$(k-1)^{-(k-1)} e^{-(k-1)\left(2k - \frac{1}{4}\right) C(r'' + 4B + 4 \cdot \log k + 1)}, \quad = M \text{ say},$$

for $C(r)$ trivially is an increasing function of r.

The quantity $4B+4\cdot\log k+1$ is negligibly small compared with $r'' \geqslant B^{4k^4}+1$. From the definition

$$C(r) = k^2 r\big((\log B)(\log r)\big)^{1/2}$$

of $C(r)$ we deduce then easily that

$$M > e^{-2k(k-1)C(r'')}.$$

Hence, on writing again r for r'', the following result has been established.

THEOREM 2. *Let $a_1, \ldots, a_{k-1}$, where $k \geqslant 2$, be distinct non-vanishing integers, and let $a > 0$ be an integer satisfying $(a, a_1, \ldots, a_{k-1}) = 1$; let further $y_1, \ldots, y_k$ be integers such that $y_k \geqslant k$. Put*

$$B = a+\max(|a_1|, \ldots, |a_{k-1}|), \qquad E_1 = e^{a_1/a}, \ldots, E_{k-1} = e^{a_{k-1}/a}$$

and define $C(r)$ and $f(r)$ as in Theorem 1. If r is the smallest positive integer satisfying

$$f(r-1) \leqslant y_k < f(r),$$

then

$$r \geqslant B^{4k^4}+1$$

and

$$y_k|y_kE_1-y_1| \ldots |y_kE_{k-1}-y_{k-1}| > e^{-2k(k-1)C(r)}.$$

21. Considerations similar to those of § 15 allow to replace this estimate by one which, although less good, is more explicit.

We now assume that

$$y_k \geqslant B^{16k^4\cdot B^{16k^4}}.$$

Under the same hypothesis as in Theorem 2 it follows then that

$$y_k|y_kE_1-y_1| \ldots |y_kE_{k-1}-y_{k-1}| > y_k^{-12k^3(k-1)((\log B)(\log\log y_k)^{-1})^{1/2}}.$$

Apart from the explicit constants, this estimate is again due to Baker.

It is highly probable that the constants in Theorem 2 and in this corollary can be improved by a direct application of Lemma 3 instead of the transfer method.

MATHEMATICS DEPARTMENT
INSTITUTE OF ADVANCED STUDIES
AUSTRALIAN NATIONAL UNIVERSITY
Canberra, ACT., Australia

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