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On the transcendency of the solutions of a special class of functional equations

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Let $a(z)$ and $b(w)$ be two rational functions in z or w with algebraic coefficients, where $a(0) = 0$ and let

$$b(w, n) = \begin{cases} 1 & \text{for } n = 0 , \\ b(w+1)b(w+2) \ldots b(w+n) & \text{for } n \geq 1 . \end{cases}$$

Assume that $0 < |z| < 1$, that z is not a pole of $a\left(z^{2^n}\right)$ for $n \geq 0$, that w is neither a pole nor a zero of $b(w, n)$ for $n \geq 1$, and that the series

$$f(z, w) = \sum_{n=0}^{\infty} a\left(z^{2^n}\right) b(w, n)$$

for fixed w is a transcendental function of z. Then, if z and w are algebraic numbers, $f(z, w)$ is a transcendental number.

In several papers of almost half a century ago (Mahler [1], [2], [3]; see also Mahler [4]) I studied the transcendency of the solutions of a general class of functional equations in one or more variables. In the case of one variable, these functional equations were of the form

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$$f(z) = \frac{a_0(z)+a_1(z)f(z^g)+\ldots+a_r(z)f(z^g)^r}{b_0(z)+b_1(z)f(z^g)+\ldots+b_r(z)f(z^g)^r}$$

where $g \geq 2$ and r are integers such that $1 \leq r \leq g-1$, and the factors $a_j(z)$ and $b_j(z)$ are polynomials in z with algebraic coefficients. By way of example, the results of this work implied the transcendency of the series

$$\sum_{n=0}^{\infty} \left(\frac{z^{2^n}}{1-z^{2.2^n}}\right)^k \quad (k = 1, 2, 3, \ldots)$$

for all algebraic numbers z satisfying $0 < |z| < 1$, hence for $z = \frac{1-\sqrt{5}}{2}$ the transcendency of

$$\sum_{n=0}^{\infty} \left(F_{2^n}\right)^{-k} \quad (k = 1, 2, 3, \ldots)$$

where F_m denotes the mth Fibonacci number.

Recently, Mignotte [5] has proved that also the series

$$\sum_{n=0}^{\infty} \left(n!.F_{2^n}\right)^{-1}$$

is transcendental. His proof is based on Schmidt's deep generalisation of Roth's Theorem (Schmidt [6]), and this new result of his is *not* contained in my old theorems.

I have therefore recently extended my old method, but in the present paper I restrict myself to a special case. The new method can almost certainly be much generalised, and it would have interest to investigate such generalisations and in particular to work out the extension to an arbitrary number of variables.

I deal here only with functions $f(z, w)$ which satisfy the functional equation

$$f(z, w) = a(z) + b(w)f(z^2, w+1) .$$

Here $a(z) \not\equiv 0$ and $b(w) \not\equiv 0$ are two rational functions in z or w

with algebraic coefficients, and the method requires that $a(0) = 0$. On putting

$$b(w, 0) = 1, \quad b(w, n) = b(w+1)b(w+2) \dots b(w+n) \quad \text{for} \quad n = 1, 2, 3, \dots,$$

the functional equation has the convergent solution

$$f(z, w) = \sum_{n=0}^{\infty} a\left(z^{2^n}\right) b(w, n)$$

whenever $0 < |z| < 1$, z is not a pole of any function $a\left(z^{2^n}\right)$, and w is not a pole of any function $b(w, n)$. If w is a zero of one of the functions $b(w, n)$, the series breaks off after finitely many terms; this trivial case is therefore also excluded. Let now (z, w) be a pair of algebraic complex numbers satisfying these restrictions, and assume in addition that this pair is such that $f(\zeta, w)$ is a transcendental function of the variable ζ. Then the new method allows to prove that the function value $f(z, w)$ is transcendental. In the special case when

$$a(z) = \frac{z}{1-z^2} \quad \text{and} \quad b(w) = \frac{1}{w}$$

and when z and w have the algebraic values

$$z = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad w = 0,$$

this result immediately gives the theorem by Mignotte.

1.

Throughout this paper, z and w are two complex variables, and

$$a(z) \not\equiv 0 \quad \text{and} \quad b(w) \not\equiv 0$$

are two rational functions which, for the present, may have arbitrary complex coefficients. We define a sequence of rational functions $b(w, n)$ by

$$b(w, 0) = 1, \quad b(w, n) = b(w+1)b(w+2) \dots b(w+n) \quad \text{for} \quad n = 1, 2, 3, \dots,$$

and a function $f(z, w)$ by

$$f(z, w) = \sum_{n=0}^{\infty} a\left(z^{2^n}\right) b(w, n). \tag{1}$$

The convergence of this series will be assured by the following assumptions.

(A) *$a(0) = 0$, and z is not a pole of any one of the functions*

$$a\left(z^{2^n}\right) \quad (n = 0, 1, 2, \ldots) .$$

(B) *w is not a zero or a pole of any one of the functions*

$$b(w, n) \quad (n = 1, 2, 3, \ldots) .$$

By the first assumption, $a(z)$ can be written as a power series

$$(2) \qquad a(z) = \sum_{j=1}^{\infty} A_j z^j$$

which converges for $|z| < |\zeta|$ where ζ is a pole of $a(z)$ closest to the origin or is the point at infinity if $a(z)$ is a polynomial. From this representation it follows that if $|z| < 1$ and $n \to \infty$,

$$a\left(z^{2^n}\right) = O\left(|z|^{2^n}\right) .$$

On the other hand, by the restriction (B), $b(w, n)$ remains finite for all n , and as $n \to \infty$ does not become larger in absolute value than a constant power of n^n . It follows that the series (1) converges in a neighbourhood of $z = 0$.

Now, from (1), $f(z, w)$ satisfies for every positive integer N the functional equation

$$(3) \qquad f(z, w) = \sum_{n=0}^{N-1} a\left(z^{2^n}\right) b(w, n) + b(w, N-1) f\left(z^{2^N}, w+N\right) .$$

By means of this equation, $f(z, w)$ can be continued into the whole of $|z| < 1$ as a meromorphic function, with poles at the poles of the functions $a\left(z^{2^n}\right)$.

Since w by (B) is not a zero of one of the functions $b(w, n)$, the series (1) does not break off after finitely many terms, which would have meant that $f(z, w)$ was a rational function of z . We require a stronger restriction.

(C) *If* w *satisfies the condition* (B), *then* $f(z, w)$ *is a transcendental function of* z .

2.

By means of the series (2), it follows from (1) that

$$f(z, w) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_j z^{2^n j} . b(w, n)$$

and hence that

(4) $$f(z, w) = \sum_{j=1}^{\infty} F_j(w) z^j ,$$

where the new Taylor coefficients $F_j(w)$ are rational functions of w given by

$$F_j(w) = \sum A_r b(w, s) ,$$

with the summation extending over all pairs of integers r, s such that

$$r \geq 1 , \quad s \geq 0 , \quad 2^s r = j .$$

More generally, let k be any non-negative integer. Then $f(z, w)^k$ can be written as a power series

$$f(z, w)^k = \sum_{j=0}^{\infty} F_{jk}(w) z^j .$$

Here, for $k = 0$,

$$F_{00}(w) = 1 , \quad F_{j0}(w) = 0 \quad \text{if} \quad j \geq 1 ;$$

for $k \geq 1$ and $0 \leq j \leq k-1$,

$$F_{jk}(w) = 0 ,$$

and for $j \geq k \geq 1$,

(5) $$F_{jk}(w) = \sum A_{r_1} \ldots A_{r_k} b(w, s_1) \ldots b(w, s_k)$$

with the summation extended over all sets of $2k$ integers $r_1, \ldots, r_k$,

$s_1, \ldots, s_k$ which satisfy the conditions

(6) $r_1 \geq 1, \ldots, r_k \geq 1$, $s_1 \geq 0, \ldots, s_k \geq 0$,

$$2^{s_1}r_1 + \ldots + 2^{s_k}r_k = j$$

Thus all Taylor coefficients $F_{jk}(w)$ are rational functions of w .

It has advantages to define $F_{jk}(w)$ also for $j < 0$ by putting

(7) $$F_{jk}(w) = 0 \quad \text{if} \quad j < 0 \; .$$

3.

Next let m be a positive integer, and let

$$C = \{c_{hk}\} \quad (h, k = 0, 1, \ldots, m)$$

be a set of $(m+1)^2$ unknowns which will soon be selected.

We form the polynomial

(8) $$r(z, w) = \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk} z^h f(z, w)^k$$

in z and $f(z, w)$ with C as the set of coefficients. It can be written as a power series

(9) $$r(z, w) = \sum_{j=0}^{\infty} R_j(w) z^j \; .$$

Here, by the power series for $f(z, w)^k$,

$$r(z, w) = \sum_{h=0}^{m} \sum_{k=0}^{m} \sum_{j=0}^{\infty} c_{hk} F_{jk}(w) z^{h+j} \; ,$$

whence

(10) $$R_j(w) = \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk} F_{j-h,k}(w) \; .$$

In the sum on the right-hand side, the convention (7) is applied for $h > j$.

These formulae show that the Taylor coefficients $R_j(w)$ of $r(z, w)$ are linear forms in the $(m+1)^2$ elements of C, with coefficients that are rational functions of w.

Consider now the system of $(m+1)^2 - 1$ homogeneous linear identities

(11) $$R_j(w) \equiv 0 \quad \text{for} \quad j = 0, 1, \ldots, (m+1)^2-2 .$$

Since the number of identities is smaller than the number of unknowns c_{hk}, we can find $(m+1)^2$ polynomials

$$c_{hk} = c_{hk}(w) \quad (h, k = 0, 1, \ldots, m)$$

of w *not all identically zero* so as to satisfy all the identities (11).

Now, by hypothesis (C), $f(z, w)$ is a *transcendental* function of z. This implies that $r(z, w)$ as just chosen cannot vanish identically in z. Hence not all the coefficients $R_j(w)$ are zero identically in w. There exists thus a smallest suffix M such that

(12) $$R_M(w) \not\equiv 0 ,$$

and here necessarily

(13) $$M \geq (m+1)^2 - 1 .$$

With this definition of M,

$$r(z, w) = R_M(w)z^M + \sum_{j=M+1}^{\infty} R_j(w)z^j .$$

4.

From now on let z and w have *fixed* values where

$$0 < |z| < 1 ,$$

and where z and w satisfy the conditions (A) and (B).

Denote by N a large positive integer and by $c_1, c_2, c_3, \ldots$ positive constants which are independent of N, but may depend on z, w, and m.

The preceding formula for $r(z, w)$ implies that

$$(14)\qquad r\left(z^{2^N}, w+N\right) = R_M(w+N)z^{2^N M} + \sum_{j=M+1}^{\infty} R_j(w+N)z^{2^N j} .$$

Here, by (10),

$$(15)\qquad R_j(w+N) = \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk}(w+N)F_{j-h,k}(w+N) ,$$

and by (5),

$$(16)\qquad F_{jk}(w+N) = \sum A_{r_1} \ldots A_{r_k} b\left(w+N, s_1\right) \ldots b\left(w+N, s_k\right) ,$$

where the summation is as in (6).

Since $a(z)$ is regular in a certain neighbourhood of $z = 0$ and vanishes at this point, there exists a positive constant c_1 independent of z, w, and m such that the Taylor coefficients A_j in (2) satisfy the inequalities

$$(17)\qquad |A_j| \le c_1^j \quad (j = 1, 2, 3, \ldots) .$$

The summation conditions (6) imply that in (16),

$$(18)\qquad r_1 + \ldots + r_k \le j \quad \text{and} \quad \max\left(s_1, \ldots, s_k\right) \le J-1$$

where J is the function of j defined by

$$J = \left[\frac{\log j}{\log 2}\right] + 1 .$$

Further the number of terms in (16) does not exceed

$$(jJ)^k ,$$

because each of the suffices $r_1, \ldots, r_k$ has at most j possibilites and each of the suffices $s_1, \ldots, s_k$ at most J.

These properties enable us to determine an upper estimate for the right-hand side of (16). Firstly, by (17) and (18),

$$\left|A_{r_1} \ldots A_{r_k}\right| \le c_1^j .$$

Next, there evidently exist two positive constants c_2 and c_3 such that for all $n \geq 0$ and for all sufficiently large positive integers N,

$$|b(w+N+n)| \leq c_2(N+n)^{c_3} .$$

Since

$$b(w+N,\ n) = b(w+N+1)b(w+N+2)\ \ldots\ b(w+N+n) ,$$

this means that

$$|b(w+N,\ n)| \leq c_2^n(N+n)^{c_3 n} ,$$

and since in (16) all the integers $s_1, \ldots, s_k$ are less than J, that

$$\left|b\left(w+N,\ s_1\right)\ \ldots\ b\left(w+N,\ s_k\right)\right| \leq c_2^{kJ}(N+J)^{c_3 kJ} .$$

It follows therefore from (16) that

$$(19) \qquad |F_{jk}(w+N)| \leq (jJ)^k . c_1^j . c_2^{kJ}(N+J)^{c_3 kJ} .$$

Since (16) was proved under the restriction that $j \geq k \geq 1$, to begin with the same restriction holds for this estimate. But in fact it holds also in the excluded cases since then $F_{jk}(w+N)$ is either 0 or 1.

5.

An upper estimate for the coefficients $R_j(w+N)$ in (14) is now easily obtained. In the formulae (15) the coefficients $c_{hk}(w+N)$ are fixed polynomials in $w + N$ which depend only on m. Hence two further positive constants c_4 and c_5 exist such that for all sufficiently large N,

$$|c_{hk}(w+N)| \leq c_4 N^{c_5} \quad (h,\ k = 0,\ 1,\ \ldots,\ m) .$$

On combining this with the formulae (15) and (19) it follows then that

$$(20) \qquad |R_j(w+N)| \leq (m+1)^2 . c_4 N^{c_5} . (jJ)^m c_1^j c_2^{mJ}(N+J)^{c_3 mJ} .$$

This upper bound will be used only for suffices j at least equal to M, hence, by (13), not less than 3. For such values of j, J is at least 2, so that

$$N+J \leq NJ ,$$

$$(N+J)^{c_3^{mJ}} \leq N^{c_3^{mJ}} J^{c_3^{mJ}} .$$

Further, by definition,

$$J = O(\log j) ,$$

and therefore

$$J^{c_3^{mJ}} = O\left(c_6^j\right) .$$

Hence (20) can be replaced by the simpler estimate

$$(21) \qquad \left|R_j(w+N)\right| \leq c_6^j N^{c_7 \log j} \quad \text{if} \quad j \geq M$$

for all sufficiently large N; here c_6 and c_7 are two further positive constants.

In particular, the rational function $R_M(w+N)$ of $w + N$ is known not to be identically zero. It follows that there exist another pair of positive constants c_8 and c_9 such that also

$$(22) \qquad N^{-c_8} \leq \left|R_M(w+N)\right| \leq N^{+c_9}$$

for all sufficiently large N.

6.

We apply now the estimates (21) and (22) to the successive terms on the right-hand side of (14). It follows immediately that

$$\left|\sum_{j=M+1}^{\infty} R_j(w+N) z^{2^N j}\right| \leq \sum_{j=M+1}^{\infty} c_6^j N^{c_7 \log j} |z|^{2^N j} ,$$

whence, on replacing j by $M + 1 + n$,

$$\left|\sum_{j=M+1}^{\infty} R_j(w+N)z^{2^N j}\right| \le$$

$$\le c_6^{M+1} N^{c_7 \log(M+1)} |z|^{2^N(M+1)} \sum_{n=0}^{\infty} c_6^n N^{c_7 \log\left(1+\left(n/(M+1)\right)\right)} |z|^{2^N n} .$$

Here

$$\log\left(1+\left(n/(M+1)\right)\right) \le \frac{n}{M+1} ,$$

and therefore

$$\sum_{n=0}^{\infty} c_6^n N^{c_7 n/(M+1)} |z|^{2^N n} = \sum_{n=0}^{\infty} \left(c_6 N^{c_7/(M+1)} |z|^{2^N}\right)^n \le 2$$

as soon as N is so large that

$$c_6 N^{c_7/(M+1)} |z|^{2^N} \le \frac{1}{2} .$$

Now increase N still further such that also

$$2c_6^{M+1} N^{c_7 \log(M+1)} |z|^{2^N M} \le \frac{1}{2} N^{-c_8} \le \frac{1}{2} N^{+c_9} .$$

Then from (14) and from the last estimates,

$$(23) \qquad \frac{1}{2} \left|R_M(w+N)z^{2^N M}\right| \le \left|r\left(z^{2^N}, w+N\right)\right| \le \frac{3}{2} \left|R_M(w+N)z^{2^N M}\right| ,$$

provided N is sufficiently large.

In this formula,

$$M \ge (m+1)^2 - 1 = m^2 + 2m \ge m^2 + 2 .$$

Further by (22), for all sufficiently large N ,

$$\left|R_M(w+N)z^{2 \cdot 2^N}\right| \le N^{c_9} |z|^{2 \cdot 2^N} \le \frac{2}{3} ,$$

and therefore

$$\left|R_M(w+N)z^{2^N M}\right| \le |z|^{m^2 \cdot 2^N} . N^{c_9} |z|^{2 \cdot 2^N} \le \frac{2}{3} |z|^{m^2 \cdot 2^N} .$$

Hence the inequality (23) leads to the following basic estimate.

There exists a positive integer N_0 *depending on* z, w, *and* m, *such that*

$$0 < \left| r\left(z^{2^N}, w+N\right) \right| \leq |z|^{m^2 . 2^N} \quad \text{for} \quad N \geq N_0 . \tag{24}$$

7.

So far, the method used was analytical, and

$$z, w, \text{ and } f = f(z, w)$$

could assume arbitrary complex values. From now on we add an arithmetical restriction and make use of number-theoretical ideas.

The new restriction is as follows.

(D) z, w, f, *and all the coefficients of* $a(z)$ *and* $b(w)$ *are algebraic numbers.*

Now each of the rational functions $a(z)$ and $b(w)$ has only finitely many coefficients. Therefore (D) requires only that a certain finite set of numbers are algebraic. Hence the operation of adjoining all the numbers of this set to the rational number field Q is equivalent to a simple algebraic extension of Q and leads to a certain algebraic number field K, say of the finite degree d over Q. Denote by O the ring of all *algebraic integers* in K.

For every element α of K let

$$\alpha = \alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(d-1)}$$

be the set of its real or complex algebraic conjugates over Q, and as usual put

$$\overline{|\alpha|} = \max_{j=0,1,\ldots,d-1} \left|\alpha^{(j)}\right| .$$

8.

The functional equation (3) can be solved for $f\left(z^{2^N}, w+N\right)$ and allows us to express this function value in terms of $f = f(z, w)$ as

$$f\left(z^{2^N}, w+N\right) = b(w, N-1)^{-1}\left\{f - \sum_{n=0}^{N-1} a\left(z^{2^N}\right)b(w, n)\right\} .$$

We combine this formula with the definition (8) of $r(z, w)$, but replace in the latter z by z^{2^N} and w by $w + N$. Since c_{hk} in (8) are now polynomials $c_{hk}(w)$ of w , it follows that

$$(25) \quad r\left(z^{2^N}, w+N\right) = \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk}(w+N)z^{2^N h}b(w, N-1)^{-k} \times$$
$$\times \left\{f - \sum_{n=0}^{N-1} a\left(z^{2^n}\right)b(w, n)\right\}^k .$$

This representation together with the new arithmetic assumption (D) allow to establish a lower estimate for $\left|r\left(z^{2^N}, w+N\right)\right|$ which by a suitable choice of m and N can be made *larger* than the upper estimate in (24), so giving a contradiction.

For this purpose we first replace (25) by an equivalent formula in which the rational functions that occur have been replaced by polynomials, all with coefficients in O .

Since $a(z)$ and $b(w)$ lie in $K(z)$ and $K(w)$, respectively, these rational functions can be written as the quotients

$$a(z) = \frac{a'(z)}{a''(z)} \quad \text{and} \quad b(w) = \frac{b'(w)}{b''(w)}$$

of polynomials in z and w , respectively, with coefficients in O . Denote by A the maximum of the degrees of $a'(z)$ and $a''(z)$, and similarly by B the maximum of the degrees of $b'(w)$ and $b''(w)$. Further put

$$a''(z, N) = \prod_{n=0}^{N-1} a''\left(z^{2^n}\right) ,$$

and

$$b'(w, 0) = 1 , \quad b'(w, n) = b'(w+1)b'(w+2) \ldots b'(w+n)$$
$$\text{for } n = 1, 2, 3, \ldots ,$$

$$b''(w,\ 0) = 1\ ,\quad b''(w,\ n) = b''(w+1)b''(w+2)\ \dots\ b''(w+n)$$
$$\text{for}\quad n = 1,\ 2,\ 3,\ \dots\ ,$$

so that for all $n \geq 0$,

$$b(w,\ n) = \frac{b'(w,n)}{b''(w,n)}\ .$$

These definitions mean that

$$a''\left(z^{2^n}\right)\quad \text{is a factor of}\quad a''(z,\ N)\quad \text{for}\quad 0 \leq n \leq N-1\ ,$$

and similarly that

$$b''(w,\ n)\quad \text{is a factor of}\quad b''(w,\ N-1)\quad \text{for}\quad 0 \leq n \leq N-1\ .$$

Further, by the hypotheses (A) and (B) all the values $a''(z,\ N)$ and $b''(w,\ N-1)$ are distinct from zero.

In this new notation, the formula (25) is now equivalent to

$$(26)\quad a''(z,\ N)^m b'(w,\ N-1)^m r\left(z^{2^N},\ w+N\right) =$$
$$= \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk}(w+N) z^{2^N h} a''(z,\ N)^{m-k} b'(w,\ N-1)^{m-k}\ \times$$
$$\times \left(a''(z,\ N)b'(w,\ N-1).f - \sum_{n=0}^{N-1} a'\left(z^{2^n}\right) \frac{a''(z,N)}{a''\left(z^{2^n}\right)}\, b'(w,\ n)\, \frac{b''(w,N-1)}{b'(w,n)}\right)^k\ .$$

We have not yet made any statement about the coefficients of the polynomials $c_{hk}(w+N)$ of $w + N$ that occur in this relation. Now, since $a(z)$ belongs to $K(z)$, its Taylor coefficients A_j lie in K . Therefore, by (5), the coefficients of the rational functions $F_{jk}(w)$ and so in particular those of the rational functions $F_{j-h,k}(w)$ that occur in the system of homogeneous linear identities (11) for the polynomials $c_{hk}(w)$ are elements of K . We are thus allowed to assume that the coefficients of these polynomials $c_{hk}(w)$ and hence also those of the polynomials

$$(27)\qquad c_{hk}(w+N)\quad (h,\ k = 0,\ 1,\ \dots,\ m)$$

lie in O, just as the coefficients of $a'(z)$, $a''(z)$, $b'(w)$, and $b''(w)$.

For shortness denote by C the maximum of the degrees of the polynomials (27); this number C depends on m, but not on N.

9.

Since the three numbers z, w, and f are elements of K, there exists a smallest positive rational integer D such that the products

$$Dz, \; Dw, \text{ and } Df$$

are algebraic integers in O.

With this definition of D, all the factors in (26) become elements of K with denominators that divide certain integral powers of D. Upper estimates for these powers are tabulated in the following table; the numbers on the right-hand side are the exponents of D.

z^{2^n}	2^n
$a'(z)$	A
$a'\left(z^{2^n}\right)$	$A.2^n$
$a''(z)$	A
$a''(z, N)$	$A\left(1 + 2 + 2^2 + \ldots + 2^{N-1}\right) = A\left(2^N-1\right)$
$\dfrac{a''(z,N)}{a''\left(z^{2^n}\right)}$ where $0 \le n \le N-1$	$A\left(2^N-1\right)$
$b'(w)$	B
$b'(w, n)$	Bn
$b'(w, N-1)$	$B(N-1)$
$b''(w)$	B
$b''(w, n)$	Bn
$b''(w, N-1)$	$B(N-1)$
$\dfrac{b''(w,N-1)}{b''(w,n)}$ where $0 \le n \le N-1$	$B(N-1)$

$c_{hk}(w+N)$	C
$a''(z,\ N)^m b'(w,\ N-1)^m$	$Am\left(2^N-1\right) + Bm(N-1)$
$c_{hk}(w+N)z^{2^N h} a''(z,\ N)^{m-k} b'(w,\ N-1)^{m-k}$ for $h,\ k = 0,\ 1,\ \ldots,\ m$	$2^N m + Am\left(2^N-1\right) + Bm(N-1) + C$
$a''(z,\ N)b'(w,\ N-1).f$	$A\left(2^N-1\right) + B(N-1) + 1 \quad 2A.2^N + 2BN$
$a'\left(z^{2^n}\right) \dfrac{a''(z,N)}{a''\left(z^{2^n}\right)} b'(w,\ n) \dfrac{b''(w,N-1)}{b''(w,n)}$ for $0 \le n \le N-1$	$A.2^N + A\left(2^N-1\right) + B(N-1) + B(N-1)$ $2A.2^N + 2BN$

By the last two lines,

$$D^{2A.2^N+2BN}.\left\{a''(z,\ N)b'(w,\ N-1).f - \sum_{n=0}^{N-1} a'\left(z^{2^n}\right) \frac{a''(z,N)}{a''\left(z^{2^n}\right)} b'(w,\ n) \frac{b''(w,N-1)}{b''(w,n)}\right\}$$

is an algebraic integer in O .

Further for $h,\ k = 0,\ 1,\ \ldots,\ m$,

$$\left(2^N m+Am\left(2^N-1\right)+Bm(N-1)+C\right) + k\left(2A.2^N+2BN\right) < 5Am.2^N$$

for all sufficiently large N , and then naturally also

$$Am\left(2^N-1\right) + Bm(N-1) < 5Am.2^N .$$

It follows therefore that the two expressions

$$(27)\quad r' = D^{5Am.2^N} . \sum_{h=0}^{m} \sum_{k=0}^{m} c_{hk}(w+N)z^{2^N h} a''(z,\ N)^{m-k} b'(w,\ N-1)^{m-k} \times$$

$$\times \left\{a''(z,\ N)b'(w,\ N-1).f - \sum_{n=0}^{N-1} a'\left(z^{2^n}\right) \frac{a''(z,N)}{a''\left(z^{2^n}\right)} b'(w,\ n) \frac{b''(w,N-1)}{b''(w,n)}\right\}^k$$

and

$$(28)\qquad r'' = D^{5Am.2^N}.a''(z,\ N)^m b'(w,\ N-1)^m$$

are algebraic integers in O and that

(29) $$r\left(z^{2^N}, w+N\right) = \frac{r'}{r''} .$$

Here the assumptions (A) and (B) ensure that

$$r'' \neq 0 .$$

10.

As before, we are only concerned with the case when z, w, and m remain fixed, while N becomes very large. But now z and w are elements of K, and therefore the new letters $c_{10}, c_{11}, c_{12}, \ldots$ will denote positive constants which depend on the d algebraic conjugates of z, the d algebraic conjugates of w, and on m, but which still are independent of N. In particular, c_{10} and c_{11} are the constants at least equal to 1 which are defined by

$$c_{10} = \max(1, \overline{|z|}) \quad \text{and} \quad c_{11} = \max(1, \overline{|w|}) .$$

If x is any element of K and $p(x)$ any polynomial in $K[x]$, then $p(x)^{(j)}$ denotes the jth conjugate of the value $p(x)$; it is obtained by replacing x by $x^{(j)}$ and all the coefficients of p by their jth conjugate. Hence

$$\overline{|p(x)|} = \max_{0 \leq j \leq d-1} |p(x)^{(j)}| .$$

In this notation, the following estimates are easily obtained. In them, j runs from 0 to $d-1$ and n from 0 to $N-1$.

$$\left|z^{(j)2^n}\right| \leq c_{10}^{2^n} .$$

$$|w^{(j)}+N| \leq 2N \quad \text{if} \quad N \geq c_{11} .$$

$$\overline{\left|a'\left(z^{2^n}\right)\right|} \leq c_{12} \cdot c_{10}^{A \cdot 2^n} .$$

$$\overline{\left|a''\left(z^{2^n}\right)\right|} \leq c_{13} \cdot c_{10}^{A \cdot 2^n} .$$

$$\overline{|a''(z,\ N)|} \le c_{13}^{N}.c_{10}^{A\left(2^{N}-1\right)} .$$

$$\overline{\left|\frac{a''(z,N)}{a''\left(z^{2^{n}}\right)}\right|} \le c_{13}^{N}.c_{10}^{A\left(2^{N}-1\right)} .$$

$$\left.\begin{aligned}
\overline{|b'(w+n)|} &\le c_{14}(2N)^{B}\\
\overline{|b'(w,\ n)|} &\le c_{14}^{n}(2N)^{Bn}\\
\overline{|b'(w,\ N-1)|} &\le c_{14}^{N-1}(2N)^{B(N-1)}\\
\overline{|b''(w+n)|} &\le c_{15}(2N)^{B}\\
\overline{|b''(w,\ n)|} &\le c_{15}^{n}(2N)^{Bn}\\
\overline{|b''(w,\ N-1)|} &\le c_{15}^{N-1}(2N)^{B(N-1)}\\
\overline{\left|\frac{b''(w,N-1)}{b''(w,n)}\right|} &\le c_{15}^{N-1}(2N)^{B(N-1)}
\end{aligned}\right\}\quad \text{if}\quad N \ge c_{11} .$$

$$\overline{|c_{hk}(w+n)|} \le c_{16}(2N)^{C} \quad \text{if} \quad N \ge c_{11} .$$

$$\overline{|f|} \le c_{17} .$$

Hence, for all sufficiently large N, by (27),

$$\overline{|r'|} \le D^{5Am.2^{N}}.(m+1)^{2}\Bigg\{c_{16}(2N)^{C}.c_{10}^{m.2^{N}}.c_{13}^{mN}c_{10}^{Am\left(2^{N}-1\right)}.c_{14}^{Bm(N-1)} \times$$

$$\times \Bigg[c_{13}^{N}c_{10}^{A\left(2^{N}-1\right)}.c_{14}^{N-1}(2N)^{B(N-1)}.c_{17} +$$

$$+ N\left(c_{12}c_{10}^{A.2^{N-1}}.c_{13}^{A\left(2^{N}-1\right)}.c_{14}^{N-1}(2N)^{B(N-1)}.c_{15}^{N-1}(2N)^{B(N-1)}\right)^{m}\Bigg]\Bigg\}$$

and

$$\overline{|r''|} \le D^{5Am.2^{N}}.c_{13}^{mN}c_{10}^{Am\left(2^{N}-1\right)}.c_{14}^{m(N-1)}(2N)^{Bm(N-1)} .$$

In these estimates the constant c_{10} does not depend on m or N. An inspection of the right-hand sides shows therefore that there exists a positive constant E independent of both m and N such that for all sufficiently large N,

(30) $$\lceil r' \rceil \le E^{m.2^N} \text{ and } \lceil r'' \rceil \le E^{m.2^N} .$$

11.

Since r' is an algebraic integer in O which does not vanish, its norm

$$\prod_{j=0}^{d-1} r'^{(j)}$$

is a rational integer distinct from 0 and so has at least the absolute value 1 . Therefore, by (3.1),

$$|r'| \ge \lceil r' \rceil^{-(d-1)} \ge E^{-(d-1)m.2^N} ,$$

while on the other hand

$$|r''| \le E^{m.2^N} .$$

It follows therefore finally from (29) that

(31) $$\left| r\left(z^{2^N}, w+N\right) \right| \ge E^{-dm.2^N}$$

for all sufficiently large N .

In the opposite direction we found already that

(24) $$\left| r\left(z^{2^N}, w+N\right) \right| \le |z|^{m^2.2^N} \text{ for } N \ge N_0$$

where $0 < |z| < 1$. Here m was up to now a fixed but otherwise arbitrary positive integer. We are thus allowed to assume that m is so large that

$$|z|^m < E^{-d} .$$

Then

$$|z|^{m^2.2^N} < E^{-dm.2^N} ,$$

and hence the two estimates (24) and (31) contradict each other.

This contradiction proves that the four hypotheses (A), (B), (C), and (D) cannot all hold simultaneously. Hence we have proved the following result.

THEOREM 1. *Let* $a(z) \not\equiv 0$ *and* $b(w) \not\equiv 0$ *be two rational functions in* z *and* w *with algebraic coefficients where*

$$a(0) = 0 .$$

Put

$$b(w,\ 0) = 1 ,\quad b(w,\ n) = b(w+1)b(w+2)\ \ldots\ b(w+n) \quad \text{for} \quad n \geq 1$$

and

$$f(z,\ w) = \sum_{n=0}^{\infty} a\left(z^{2^n}\right) b(w,\ n) .$$

Assume that if $0 < |z| < 1$, *if* z *is not a pole of any one of the functions* $a\left(z^{2^n}\right)$ *where* $n \geq 0$, *and if* w *is neither a zero nor a pole of any one of the functions* $b(w,\ n)$ *where* $n \geq 1$, *then* $f(z,\ w)$ *is a transcendental function of* z .

Then, if z *and* w *still satisfy these restrictions and in addition are algebraic numbers, the function value* $f(z,\ w)$ *is transcendental.*

12.

Let us consider one example. Choose

$$a(z) = \frac{z}{1-z^2} ,$$

and take

$$b(w) = \prod_{\rho=1}^{r} \left(w+\alpha_\rho-1\right) \cdot \prod_{\sigma=1}^{s} \left(w+\beta_\sigma-1\right)^{-1}$$

where r and s are arbitrary non-negative integers, and the constants α_ρ and β_σ are algebraic numbers which, for reasons that will soon become clear, are assumed to be real. Since in terms of the Gamma function,

$$b(w) = \prod_{\rho=1}^{r} \frac{\Gamma\left(w+\alpha_\rho\right)}{\Gamma\left(w+\alpha_\rho-1\right)} \cdot \prod_{\sigma=1}^{s} \frac{\Gamma\left(w+\beta_\sigma-1\right)}{\Gamma\left(w+\beta_\sigma\right)} ,$$

it follows that for $n \geq 0$,

$$b(w,\ n) = \prod_{\rho=1}^{r} \frac{\Gamma\left(w+\alpha_{\rho}+n\right)}{\Gamma\left(w+\alpha_{\rho}\right)} \cdot \prod_{\sigma=1}^{s} \frac{\Gamma\left(w+\beta_{\sigma}\right)}{\Gamma\left(w+\beta_{\sigma}+n\right)} .$$

Here let w be fixed and not a zero or pole of $b(w,\ n)$ for any $n \geq 0$. Then for large n the value of $b(w,\ n)$ is real if w is real, and it has a fixed sign. This means that, if $0 < z < 1$, then all terms of the series $f(z,\ w)$ have finally the same sign, and hence $f(z,\ w)$ tends to plus or minus infinity as z tends to 1 . But then, from the form of the series, the same is true if z tends radially to a 2^kth root of unity for any positive integer k . These roots of unity lie dense on the unit circle and so this circle is a natural boundary for $f(z,\ w)$, and hence $f(z,\ w)$ is a transcendental function of z .

When the numbers w, α_{ρ} , and β_{σ} are not all real, this simple proof breaks down, and there may possibly be cases when $f(z,\ w)$ becomes rational in z .

In any case, on putting w equal to zero which now is not an essential restriction, we obtain the following result.

THEOREM 2. *Let* r *and* s *be non-negative integers, and let* $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ *be real algebraic numbers which are all distinct from* 0 *and the negative integers; let further* z *be any algebraic number satisfying*

$$0 < |z| < 1 .$$

Then the infinite series

$$\sum_{n=0}^{\infty} \sum_{\rho=1}^{r} \frac{\Gamma\left(\alpha_{\rho}+n\right)}{\Gamma\left(\alpha_{\rho}\right)} \cdot \prod_{\sigma=1}^{s} \frac{\Gamma\left(\beta_{\sigma}\right)}{\Gamma\left(\beta_{\sigma}+n\right)} \cdot \frac{z^{2^n}}{1-z^{2\cdot 2^n}}$$

is a transcendental number.

By way of example, let us choose

$$z = \frac{1-\sqrt{5}}{2} , \text{ so that } -1/z = \frac{1+\sqrt{5}}{2} .$$

Then, for $n \geq 1$,

$$\sqrt{5}\,\frac{z^{2^n}}{1-z^{2.2^n}} = \left(F_{2^n}\right)^{-1}$$

where F_m denotes the mth Fibonacci number. Hence Theorem 2 implies the transcendency of the series

$$\sum_{n=0}^{\infty} \prod_{\rho=1}^{r} \frac{\Gamma(\alpha_\rho+n)}{\Gamma(\alpha_\rho)} \cdot \prod_{\sigma=1}^{s} \frac{\Gamma(\beta_\sigma)}{\Gamma(\beta_\sigma+n)} \cdot \left(F_{2^n}\right)^{-1} .$$

In the special case when $r = 0$, $s = 1$, $\beta_1 = 1$, this result is that by Mignotte referred to in the introduction.

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