ON A CLASS OF NON-LINEAR FUNCTIONAL EQUATIONS CONNECTED WITH MODULAR FUNCTIONS

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ON A CLASS OF NON-LINEAR FUNCTIONAL EQUATIONS CONNECTED WITH MODULAR FUNCTIONS

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Abstract

Let p be a prime. This paper deals with solutions of functional equations

$$f(z^p)^{p+1} + f(z)^{p+1} + \sum_{r=0}^{p}\sum_{s=0}^{p} c_{rs} f(z^p)^r f(z)^s = 0 \qquad (c_{rs} = c_{sr})$$

in either formal Laurent series or in analytic functions. Examples connected to special modular functions are considered.

Many years ago, in several papers dealing mainly with questions of transcendency, I have studied (Mahler 1929, 1930a, 1930b) functional equations of the type

$$f(z^n) = R(z, f(z)),$$

where $n \geqq 2$ is a fixed integer, and $R(z, w)$ is a rational function of its arguments the numerator and the denominator of which are at most of degree $n-1$ in w.

From the theory of transformation equations of modular functions one is led to study the related type of functional equations

$$\text{(E)} \qquad f(z^n)^{n+1} + f(z)^{n+1} + \sum_{r=0}^{n}\sum_{s=0}^{n} c_{rs} f(z^n)^r f(z)^s = 0$$

where the coefficients c_{rs} are constants satisfying the symmetry conditions

$$\text{(S)} \qquad c_{rs} = c_{sr} \quad \text{for all } r, s.$$

Here the unknown function $f(z)$ is assumed to be analytic with at most a pole at $z = 0$.

Although this kind of functional equation can be studied for any integer $n \geqq 2$, the theory becomes particularly simple when $n = p$ is a prime. It is therefore this special class of functional equations which will be considered in the present paper. It would have some interest to investigate also the case of general n, and further to study the more general class of functional equations where the symmetry conditions (S) are not satisfied.

The first two chapters deal with solutions of the equation (E) that are *formal* Laurent series $f(z)$ in an indeterminate z, and we determine in particular the number of independent parameters on which the set of all possible solutions $f(z)$ of this kind may depend. The third chapter contains then a proof that the formal solutions $f(z)$ do in fact converge for $0 < |z| < \gamma$ where γ denotes a sufficiently small positive constant. This allows to prove that $f(z)$ may be continued into the whole unit circle, but may then possess an infinity of algebraic branch points.

In the last chapter we finally discuss connections of our results to special modular functions and some of their transformation equations. Here the group properties of the modular functions play only a very implicit role.

1. The basic *modular function* $j(\omega)$ *of level* 1 is invariant under the full *modular group* of all linear substitutions

$$\omega \rightarrow \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha, \beta, \gamma, \delta \text{ integers satisfying } \alpha\delta - \beta\gamma = 1).$$

The function $j(\omega)$ is regular in the upper halfplane $\mathrm{Im}(\omega) > 0$ and has a simple pole at infinity which is made evident by the convergent Fourier expansion

$$j(\omega) = \sum_{h=-1}^{\infty} a_h e^{2\pi i h \omega} \quad \text{where} \quad a_{-1} = 1, \; a_0 = 744, \text{ etc.}$$

Of fundamental importance are the *transformation equations for* $j(\omega)$. Let $n \geqq 2$ be any integer. There exists then a unique irreducible symmetric polynomial

$$F_n(X, Y) = F_n(Y, X)$$

of the exact degree

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

in both X and Y, with the highest terms

$$X^{\psi(n)} \quad \text{and} \quad Y^{\psi(n)},$$

such that

$$F_n(j(n\omega), j(\omega)) \equiv 0 \quad \text{identically in } \omega.$$

More exactly, if $\{A, B, D\}$ runs over all $\psi(n)$ triplets of integers satisfying

$$A > 0, \quad 0 \leqq B \leqq D - 1, \quad D > 0, \quad AD = n, \quad (A, B, D) = 1,$$

then the equation

$$F_n(X, j(\omega)) = 0$$

has as its roots exactly all the $\psi(n)$ transformed functions

$$j\left(\frac{A\omega + B}{D}\right).$$

Of particular interest for us is the special case when $n = p$ is a prime and therefore $\psi(p) = p + 1$. Now the transformation equation

$$F_p(X, j(\omega)) = 0$$

has the $p + 1$ roots

$$j(p\omega), j\left(\frac{\omega}{p}\right), j\left(\frac{\omega+1}{p}\right), \cdots, j\left(\frac{\omega+p-1}{p}\right),$$

and the transformation polynomial $F_p(X, Y)$ has the explicit form

$$F_p(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^r Y^s$$

where the coefficients c_{rs} are certain integers divisible by p and satisfying

$$c_{rs} = c_{sr}, \qquad c_{pp} = 0.$$

2. Put

$$z = e^{2\pi i\omega} \quad \text{and} \quad j(\omega) = f(z),$$

so that

$$j(\omega) = f(z) = \sum_{h=-1}^{\infty} a_h z^h$$

becomes a Laurent series in the integral powers of z which converges for

$$0 < |z| < 1.$$

(The new variable z corresponds to the quantities q^2 of Jacobi and r of Klein.) In terms of $f(z)$, the roots of the transformation equation

$$F_p(X, f(z)) = 0$$

can be written as

$$f(z^p), f(z^{1/p}), f(\varepsilon z^{1/p}), \cdots, f(\varepsilon^{p-1} z^{1/p}),$$

where

$$\varepsilon = e^{2\pi i/p}$$

is a primitive pth root of unity.

If $F_p(X, f(z))$ is written in its explicit form as a polynomial in X,

$$F_p(X, f(z)) = X^{p+1} - s_1 X^p + s_2 X^{p-1} - + \cdots \pm s_{p+1},$$

then the coefficients $s_1, s_2, \cdots, s_{p+1}$ are the elementary symmetric functions of its roots,

$$\begin{aligned} s_1 &= f(z^p) + f(z^{1/p}) + f(\varepsilon z^{1/p}) + \cdots + f(\varepsilon^{p-1} z^{1/p}), \\ s_2 &= f(z^p)f(z^{1/p}) + f(z^p)f(\varepsilon z^{1/p}) + \cdots + f(\varepsilon^{p-2} z^{1/p})f(\varepsilon^{p-1} z^{1/p}), \\ &\cdot \\ &\cdot \\ &\cdot \\ s_{p+1} &= f(z^p)f(z^{1/p})f(\varepsilon z^{1/p}) \cdots f(\varepsilon^{p-1} z^{1/p}). \end{aligned} \tag{1}$$

Each of the elementary symmetric functions is a polynomial in $f(z)$ with constant (integral) coefficients, and while $s_1, s_2, \cdots, s_p$ are at most of degree p in $f(z)$, s_{p+1} has the exact degree $p + 1$ in $f(z)$.

3. The properties just quoted have their analogues for other modular functions, including those of higher level. In this paper, we shall try to extend several of these properties to the solutions of a more general class of functional equations

$$F(f(z^p), f(z)) = 0,$$

where $F(X, Y)$ is a symmetric polynomial in X and Y of degree $p + 1$. The more general kind of functional equation

$$F(f(z^n), f(z)) = 0,$$

where $n \geqq 2$ is a composite number, can be treated similarly, but requires the discussion of more cases.

Chapter 1. **Formal Laurent Series**

4. For the present we shall not be concerned with analytic functions of ω or z, but rather with formal Laurent series in one indeterminate z.

Denote by p a fixed prime and by K an arbitrary field not of characteristic p which contains a root $\varepsilon \neq 1$ of the equation $x^p - 1 = 0$; its p roots $1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{p-1}$ are then all distinct. Elements of K will be called *constants*.

Next let z be an indeterminate over K, and let $K[[z]]$ denote the field of all formal ascending Laurent series

$$f(z) = \sum_{h=m}^{\infty} a_h z^h$$

with coefficients a_h in K. If the trivial case $f(z) \equiv 0$ is excluded, the notation can always be chosen such that $a_m \neq 0$. Then m is called the *order* of $f(z)$; this order may be negative, zero or positive.

With the series $f(z)$ there can be associated the set, Σ_f say, of the $p+1$ derived Laurent series

$$\Sigma_f: \qquad f(z^p); f(z^{1/p}), f(\varepsilon z^{1/p}), f(\varepsilon^2 z^{1/p}), \cdots, f(\varepsilon^{p-1} z^{1/p})$$

in z^p, and $z^{1/p}$, respectively, and we can further form the elementary symmetric functions $s_1, s_2, \cdots, s_{p+1}$ of these series, as defined in (1).

We now say that $f(z)$ *is an* S_p*-series if*

(A) *No two of the elements of* Σ_f *are identical*; and
(B) *Every elementary symmetric function* $s_1, s_2, \cdots, s_{p+1}$ *of the elements of* Σ_f *can be expressed as a polynomial in* $f(z)$ *with coefficients in* K.

5. A number of simple properties of such S_p-series $f(z)$ follow immediately from this definition.

Firstly, by (A), $f(z)$ cannot be a constant since then all the elements of Σ_f would be equal to the same constant.

Secondly, since then $f(z) \notin K$, it can easily be proved that $f(z)$ is transcendental over K.

Thirdly, if $C_0 \neq 0$ and C_1 are arbitrary constants, also $C_0 f(z) + C_1$ is an S_p-series and this series has the same order as $f(z)$, unless $f(z)$ has the order 0.

Fourthly, if n is any positive integer, also $f(z^n)$ is an S_p-series, but of order mn.

Fifth, the representations of the elementary functions s_k as polynomials in $f(z)$ with coefficients in K are unique by the transcendency of $f(z)$.

The last property implies that we can associate with every S_p-series $f(z)$ a *unique* polynomial $F(X, Y)$ in two independent indeterminates X and Y by putting

$$
\begin{aligned}
F(X, f(z)) &= (X - f(z^p))\prod_{j=0}^{p-1} (X - f(\varepsilon^j z^{1/p})) \\
&= X^{p+1} - s_1 X^p + s_2 X^{p-1} - + \cdots \pm s_{p+1}.
\end{aligned}
\tag{2}
$$

This polynomial $F(X, Y)$ has again coefficients in K; we call it the *polynomial of* $f(z)$. Before studying this polynomial, we establish a general property of certain symmetric polynomials.

6. Denote by $\Phi(X, Y) = \Phi(Y, X)$ any *symmetric* polynomial in X and Y of the form

$$\Phi(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} \gamma_{rs} X^r Y^s$$

with coefficients $\gamma_{rs} = \gamma_{sr}$ in K. The following result can then be proved.

THEOREM 1. *Let $f(z)$ be a series in $K[[z]]$ such that all the elements of Σ_f are distinct. If any one of the $p+1$ equations*

$$\Phi(f(z^p), f(z)) = 0 \quad \text{and} \quad \Phi(f(\varepsilon^j z^{1/p}), f(z)) = 0, \quad \text{where} \quad j = 0, 1, \cdots, p-1,$$

is satisfied, then $f(z)$ is an S_p-series and $\Phi(X, Y)$ is its polynomial.

PROOF. It evidently suffices to prove that each of these equations implies the other p equations; for then the elementary symmetric functions s_k of the elements of Σ_f become polynomials in $f(z)$ with coefficients in K since they are the roots of the monic equation

$$\Phi(X, f(z)) = 0.$$

If, firstly,

$$\Phi(f(z^p), f(z)) = 0,$$

then, on replacing z by $\varepsilon^j z^{1/p}$, it follows that also

$$\Phi(f(z), f(\varepsilon^j z^{1/p})) = 0 \qquad (j = 0, 1, \cdots, p-1),$$

and the assertion is an immediate consequence of the symmetry of $\Phi(X, Y)$.

Secondly, let for some j

$$\Phi(f(\varepsilon^j z^{1/p}), f(z)) = 0.$$

If now $\varepsilon^j z^{1/p}$ is replaced by z, and hence $(\varepsilon^j z^{1/p})^p = z$ by z^p, we obtain the equation

$$\Phi(f(z), f(z^p)) = 0,$$

and so, by the symmetry of $\Phi(X, Y)$, we are back in the first case in which the assertion has already been proved.

7. The following theorem is basic for the further theory.

THEOREM 2. *Let $f(z)$ be any S_p-series, and let $F(X, Y)$ be the polynomial of $f(z)$. Then this polynomial is symmetric in X and Y and hence is of the form*

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^r Y^s \tag{3}$$

with coefficients c_{rs} in K satisfying the symmetry conditions

$$c_{rs} = c_{sr} \qquad (r, s = 1, \cdots, p).$$

PROOF. Since $F(X, Y)$ is monic in $\$$, it suffices to prove the symmetry of $F(X, Y)$ because it has then the asserted form and the coefficients c_{rs} satisfy $c_{rs} = c_{sr}$.

Two cases have to be distinguished. First assume that $F(X, Y)$ is irreducible over K. Since both $f(z^p)$ and $f(z^{1/p})$ lie in Σ_f, evidently

$$F(f(z^p), f(z)) = 0 \quad \text{and} \quad F(f(z^{1/p}), f(z)) = 0.$$

On replacing z by z^p in the second equation, it follows that also

$$F(f(z), f(z^p)) = 0.$$

The irreducible equation

$$F(X, f(z)) = 0 \tag{4}$$

and the equation

$$F(f(z), X) = 0$$

have therefore the root $f(z^p)$ in common and therefore share all the $p+1$ elements of Σ_f as roots. This requires, firstly, that $F(X, Y)$ is at least of degree $p+1$ in Y. Secondly, $F(Y, X)$ necessarily is divisible by $F(X, Y)$ and hence has the form

$$F(Y, X) = h(X, Y) F(X, Y) \tag{5}$$

where $h(X, Y)$ is a certain polynomial in $K[X, Y]$. Here the degree in Y of $F(Y, X)$ is exactly $p+1$, and that of $F(X, Y)$ is at least $p+1$. Hence both polynomials are exactly of the same degrees $p+1$ in both X and Y, and so the factor

$$h(X, Y) = h$$

is a constant. On applying twice the identity (5), once after interchanging X and Y, it follows that

$$F(Y, X) = hF(X, Y) = h^2F(Y, X),$$

hence that $h^2 = 1$. Here h cannot have the value -1 since then $F(Y, X) = -F(X, Y)$ and therefore $F(X, X) \equiv 0$ so that $F(X, Y)$ is divisible by $X - Y$, contrary to the irreducibility of $F(X, Y)$. Hence $h = +1$ and $F(X, Y) = F(Y, X)$, whence the assertion.

Secondly let $F(X, Y)$ be reducible. It has then a factorisation

$$F(X, Y) = G(X, Y)H(X, Y)$$

where both $G(X, Y)$ and $H(X, Y)$ are polynomials in $K[X, Y]$ which have positive degrees in X. The two equations

$$G(X, f(z)) = 0 \tag{6}$$

and

$$H(X, f(z)) = 0 \tag{7}$$

together have all the $p + 1$ elements of Σ_f as their roots. Without loss of generality the notation can be chosen such that the first equation (6) is irreducible and has $f(z^{1/p})$ as one of its roots. Replace $z^{1/p}$ in this equation (6) successively by $\varepsilon z^{1/p}, \varepsilon^2 z^{1/p}, \cdots, \varepsilon^{p-1} z^{1/p}$. These substitutions leave both z and $f(z)$ unchanged, but transform $f(z^{1/p})$ into $f(\varepsilon z^{1/p}), f(\varepsilon^2 z^{1/p}), \cdots, f(\varepsilon^{p-1} z^{1/p})$, respectively, which therefore likewise satisfy the equation (6). This equation cannot have more than p roots, and so the remaining element $f(z^p)$ of Σ_f necessarily satisfies the second equation (7). This equation (7) is linear and so is also irreducible.

It follows then that

$$G(f(z^{1/p}), f(z)) = 0 \quad \text{and} \quad H(f(z^p), f(z)) = 0,$$

hence, on replacing z by z^p and $z^{1/p}$, respectively, that also

$$G(f(z), f(z^p)) = 0 \quad \text{and} \quad H(f(z), f(z^{1/p})) = 0.$$

It has thus been shown that $f(z^{1/p})$ is a root of both irreducible equation (6) and of the equation

$$H(f(z), X) = 0,$$

and similarly that $f(z^p)$ is a root of both the irreducible equation (7) and of

$$G(f(z), X) = 0.$$

These properties require that $H(Y, X)$ be divisible by $G(X, Y)$ and $G(Y, X)$ be divisible by $H(X, Y)$, hence that $G(X, Y)$ also be divisible by $H(Y, X)$. Hence there is a constant $h \neq 0$ such that

$$G(X, Y) = hH(Y, X)$$

and therefore that

$$F(X, Y) = G(X, Y)H(X, Y) = hH(X, Y)H(Y, X).$$

This identity shows that also in the present case $F(X, Y)$ is symmetrical in X and Y.

8. Let again $f(z)$ be an S_p-series, and let

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^r Y^s \qquad (c_{rs} = c_{sr})$$

be its polynomial. We know already that if $C_0 \neq 0$ and C_1 are arbitrary constants, then also

$$g(z) = C_0 f(z) + C_1$$

is an S_p-series. Denote by

$$G(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} d_{rs} X^r Y^s, \qquad (d_{rs} = d_{sr}),$$

the polynomial of $g(z)$.

The set Σ_g consists of the $p+1$ Laurent series

$$g(z^p) = C_0 f(z^p) + C_1;$$

$$g(\varepsilon^j z^{1/p}) = C_0 f(\varepsilon^j z^{1/p}) + C_1 \quad \text{where} \quad j = 0, 1, \cdots, p-1,$$

and these series are all distinct and satisfy the equation

$$G(X, g(z)) = 0,$$

while similarly the elements of Σ_f are the roots of the equation

(4) $$F(X, f(z)) = 0.$$

Here both polynomials $F(X, Y)$ and $G(X, Y)$ are monic with respect to X and to Y. They are therefore connected by the identity

(8) $$C_0^{p+1} G(X, Y) = F(C_0 X + C_1, C_0 Y + C_1),$$

from which it easily follows that

(9a) $$C_0^{p+1} d_{00} = 2\, C_1^{p+1} + \sum_{R=0}^{p} \sum_{S=0}^{p} c_{RS}\, C_1^{R+S},$$

(9b) $$C_0^{p-s+1} d_{0s} = \binom{p+1}{s} C_1^{p-s+1} + \sum_{R=0}^{p} \sum_{S=s}^{p} c_{RS} \binom{S}{s} C_1^{R+S-s} \quad (s = 1, 2, \cdots, p),$$

(9c) $$C_0^{p-r-s+1} d_{rs} = \sum_{R=r}^{p} \sum_{S=s}^{p} c_{RS} \binom{R}{r} \binom{S}{s} C_1^{R+S-r-s} \quad (r, s = 1, 2, \cdots, p).$$

In the special case when $C_1 = 0$ and therefore

$$g(z) = C_0 f(z),$$

these formulae take the simpler form

(10) $$d_{rs} = C_0^{r+s-p-1} c_{rs} \qquad (r, s = 0, 1, \cdots, p).$$

We further note that, under no restriction on C_1, the equations (9c) imply in particular that

(11) $$d_{pp} = C_0^{p-1} c_{pp}.$$

9. There is one further set of transformation formulae which will soon be needed.

Let again $f(z)$ be any S_p-series and let $F(X, Y)$ be its polynomial. We say that $f(z)$ is an R_p-series if also its reciprocal $f(z)^{-1}$ is an S_p-series. Thus both $f(z)$ and $f(z)^{-1}$ are simultaneously S_p-series and R_p-series.

Assume that $f(z)$ is such an R_p-series, and denote as before by $F(X, Y)$ the polynomial of $f(z)$ and similarly by

$$F^0(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c^0_{rs} X^r Y^s \qquad (c^0_{rs} = c^0_{sr})$$

the polynomial of $f(z)^{-1}$. The set $\Sigma_{f^{-1}}$ consists of the $p+1$ Laurent series

$$f(z^p)^{-1}; f(z^{1/p})^{-1}, f(\varepsilon z^{1/p})^{-1}, \cdots, f(\varepsilon^{p-1} z^{1/p})^{-1},$$

and both $F(X, Y)$ and $F^0(X, Y)$ are monic and of the exact degrees $p+1$ in X and in Y. These two polynomials are therefore connected by the pair of identities

(12) $$F^0(X, Y) = X^{p+1} Y^{p+1} F(X^{-1}, Y^{-1}), \quad F(X, Y) = X^{p+1} Y^{p+1} F^0(X^{-1}, Y^{-1}).$$

The explicit formula (3) for $F(X, Y)$ implies then that $F^0(X, Y)$ is given by

$$F^0(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^{p-r+1} Y^{p-s+1},$$

and here both exponents $p-r+1$ and $p-s+1$ are at least 1. But $F^0(X, Y)$ may not contain any terms divisible by $X^{p+1}Y$ or by XY^{p+1}. Therefore necessarily

(13) $$c_{0s} = c_{s0} = 0 \qquad (s = 0, 1, \cdots, p).$$

Hence in the present case $F(X, Y)$ has the special form

(14a) $$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c_{rs}X^rY^s,$$

and similarly $F^0(X, Y)$ has the form

(14b) $$F^0(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c^0_{rs}X^rY^s$$

with the coefficients

(15) $$c^0_{rs} = c_{p-r+1,p-s+1} \qquad (r, s = 1, 2, \cdots, p).$$

By the complete symmetry in $F(X, Y)$ and $F^0(X, Y)$ we can then conclude that

The conditions (13) *are both necessary and sufficient for $f(z)$ to be an R_p-series.*

It is useful to note that both the double sums $\sum_{r=1}^{p}\sum_{s=1}^{p}$ in (14a) and (14b) are divisible by XY, hence are equal to zero if $X = 0$ or $Y = 0$.

10. The following theorem shows the special role of series of positive order.

THEOREM 3. *Every S_p-series of positive order is also an R_p-series.*

PROOF. Let $f(z)$ be any S_p-series of positive order. It suffices to show that the polynomial $F(X, Y)$ of $f(z)$ has the form (14a), i.e. that its coefficients c_{rs} satisfy the conditions (13).

Assume this assertion is not true. There exists then a suffix σ in the interval $0 \leqq \sigma \leqq p$ such that

$$c_{0s} = 0 \quad \text{for} \quad s = 0, 1, \cdots, \sigma - 1, \quad \text{but that } c_{0\sigma} \neq 0.$$

We apply the identity

$$F(f(z^p), f(z)) = f(z^p)^{p+1} + f(z)^{p+1} + \sum_{r=0}^{p}\sum_{s=0}^{p} c_{rs} f(z^p)^r f(z)^s = 0.$$

Here by hypothesis

$$f(z) = a_m z^m + \cdots \quad \text{and} \quad f(z^p) = a_m z^{mp} + \cdots$$

where $m > 0$, $a_m \neq 0$, and the dots denote terms in higher powers of z. The identity implies then that

$$(16)\quad a_m^{p+1} z^{m(p^2+p)} + \cdots + a_m^{p+1} z^{m(p+1)} + \cdots + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} (a_m^{r+s} z^{m(pr+s)} + \cdots) = 0.$$

If now $0 \leqq \sigma \leqq p-1$, then there is in this identity only one term $c_{0\sigma} a_m^{\sigma} z^{m\sigma}$ involving the lowest occurring power of z, and this term cannot be cancelled by any other term. If, however, $\sigma = p$, then $c_{10} = c_{01} = 0$ by the definition of σ, and so $c_{0p} a_m^{p} z^{mp}$, which is now the term involving the lowest power of z, again cannot be cancelled by any other term. This proves the assertion.

We shall soon show that there are S_p-series of negative order which are not also R_p-series. There is thus an essential difference between the S_p-series of positive and of negative orders.

11. The next problem to be discussed concerns the question whether, for a given integer m and a given constant $a_m \neq 0$, there exists an S_p-series

$$f(z) = \sum_{h=m}^{\infty} a_h z^h$$

of order m and with the first coefficient a_m. The answer will depend on whether m is positive, zero, or negative, and the conditions will be different in these three cases.

We begin with the easiest case when m is negative, and we apply again the identity (16) where now, however, m is a negative integer.

On writing the double sum in this identity as a Laurent series, the term in the highest occurring negative power of z evidently is

$$c_{pp} a_m^{2p} \cdot z^{m(p^2+p)}.$$

Similarly, the term in the highest occurring negative power of z arising from

$$(a_m z^{mp} + \cdots)^{p+1} + (a_m z^m + \cdots)^{p+1}$$

is equal to

$$a_m^{p+1} \cdot z^{m(p^2+p)}.$$

These two terms must cancel one another, whence there follows the necessary condition

$$(17)\qquad c_{pp} = -a_m^{-p+1}$$

for a_m. This equation shows in particular that

the coefficient c_{pp} of the polynomial $F(X, Y)$ of $f(z)$ is distinct from zero if $m < 0$, and $-c_{pp}$ is then the $(p-1)$st power of an element of K.

12. In order to derive also sufficient conditions, it is convenient to put

$$f(z) = a_m g(z), \quad \text{where} \quad g(z) = z^m + \sum_{h=m+1}^{\infty} b_h z^h$$

is again an S_p-series of order $m < 0$, but with the special property that 1 is the coefficient of z^m. Such a series is said to be *normed.*

As in §8, denote by

$$G(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} d_{rs} X^r Y^s \qquad (d_{rs} = d_{sr})$$

the polynomial of $g(z)$. By (10), this polynomial has the coefficients

(18) $$d_{rs} = a_m^{r+s-p-1} c_{rs} \qquad (r, s = 0, 1, \cdots, p).$$

In particular, the necessary condition (17) takes now the simple form

(19) $$d_{pp} = -1.$$

This formula suggests to write $G(X, Y)$ in the equivalent, but more convenient form

(20) $$G(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{r=0}^{p} \sum_{s=0}^{p} D_{rs} X^r Y^s \qquad (D_{rs} = D_{sr}),$$

where

(21) $$D_{pp} = d_{pp} + 1 = 0, \quad D_{11} = d_{11} + 1, \quad \text{and otherwise } D_{rs} = d_{rs} \qquad (r, s = 0, 1, \cdots, p).$$

Denote now by D the set of all coefficients

$$D_{rs}, \quad \text{where} \quad 0 \leqq r \leqq s \leqq p,$$

with the exception of the trivial coefficient $D_{pp} = 0$. Further, if n is any integer greater than m, let B_n be any polynomial in

$$b_{m+1}, b_{m+2}, \cdots, b_n \text{ and in the elements of } D$$

with coefficients in K. The symbol B_n need not always denote the same polynomial.

13. The following basic existence result can now be proved.

THEOREM 4. *Let $G(X, Y)$ be any symmetric polynomial of the form* (20) *with coefficients D_{rs} in K where $D_{pp} = 0$, and let m be any negative integer. Then there exists one and only one normed S_p-series $g(z)$ of order m such that $G(X, Y)$ is its polynomial.*

PROOF. In analogy to earlier proofs, we apply the identity

$$(22)\qquad \begin{aligned} &G(g(z^p), g(z)) = \\ &= (g(z^p)^p - g(z))(g(z)^p - g(z^p)) + \sum_{r=0}^{p}\sum_{s=0}^{p} D_{rs} g(z^p)^r g(z)^s = 0. \end{aligned}$$

Here

$$g(z) = z^m + \sum_{h=1}^{\infty} b_{h+m} z^{h+m}, \quad g(z^p) = z^{mp} + \sum_{h=1}^{\infty} b_{h+m} z^{(h+m)p},$$

hence

$$g(z)^s = z^{ms} + sb_{m+1} z^{ms+1} + \sum_{h=2}^{\infty} (sb_{h+m} + B_{h+m-1}) z^{ms+h},$$

$$g(z^p)^r = z^{mpr} + rb_{m+1} z^{(ms+1)p} + \sum_{h=2}^{\infty} (rb_{h+m} + B_{h+m-1}) z^{(mr+h)p}.$$

Further, since p is at least 2,

$$g(z^p)^p - g(z) = z^{mp^2} + \sum_{h=2}^{\infty} B_{h+m-1} z^{mp^2+h}$$

and

$$g(z)^p - g(z^p) = pb_{m+1} z^{pm+1} + \sum_{h=2}^{\infty} (pb_{h+m} + B_{h+m-1}) z^{mp+h},$$

whence

$$\begin{aligned} &(g(z^p)^p - g(z))(g(z)^p - g(z)) = \\ &\qquad = pb_{m+1} z^{m(p^2+p)+1} + \sum_{h=2}^{\infty} (pb_{h+m} + B_{h+m-1}) z^{m(p^2+p)+h}. \end{aligned}$$

On the other hand,

$$D_{rs} g(z^p)^r g(z)^s = D_{rs} z^{m(pr+s)} + \sum_{h=1}^{\infty} B_{h+m} z^{m(pr+s)+h}$$

$$(r, s = 0, 1, \cdots, p).$$

On substituting these Laurent series in (22) and equating the coefficients of all the different powers of z to zero, we obtain an infinite system of recursive formulae

$$(23)\qquad pb_{m+1} = D_{p,p-1}; \quad pb_{h+m} = B_{h+m-1} \quad \text{for} \quad h = 2, 3, 4, \cdots.$$

These formulae allow to determine step by step all the coefficients b_{h+m} and prove both their existence and uniqueness.

The proof implies the following result.

COROLLARY. *The coefficients b_{h+m} of $g(z)$ can be expressed as polynomials in the elements of D with coefficients in K.*

By means of a slight change of method one can show that, conversely, *all the elements of D can be written as polynomials in finitely many of the Laurent coefficients b_{h+m} with coefficients in K.* This is also implicit in the definition of $G(X, Y)$ by means of the elementary symmetric functions of the elements of Σ_g.

14. Denote by $g(z \mid m)$ the normed S_p-series of order $m < 0$ given by Theorem 4, and put

(24) $$g[z] = g(z \mid -1).$$

The identity

$$G(g[z^p], g[z]) = 0$$

implies that for every positive integer n also

$$G(g[z^{np}], g[z^n]) = 0,$$

and here, by §5, $g[z^n]$ is again a normed S_p-series, but of order $-n$. The uniqueness proved in Theorem 4 implies then the identity

(25) $$g(z \mid m) = g[z^{-m}] \qquad \textit{for every integer } m < 0.$$

Thus, once $g[z]$ is known, we know *all* the normed S_p-series of negative orders of which $G(X, Y)$ is the polynomial.

On returning to the original S_p-series $f(z)$ of order $m < 0$ with coefficient $a_m \neq 0$, Theorem 4 leads easily to the following more general result.

THEOREM 5. *Let*

(3) $$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^r Y^s$$

be any symmetric polynomial with coefficients c_{rs} in K where $c_{pp} \neq 0$, and let

$$f(z) = \sum_{h=m}^{\infty} a_h z^h$$

be any S_p-series of order $m < 0$ with the polynomial $F(X, Y)$. Then a_m necessarily is a root in K of the algebraic equation

$$a_m^{p-1} = -c_{pp}^{-1},$$

and $f(z)$ *has the form*

$$f(z) = a_m \cdot g[z^{-m}]$$

where $g[z]$ *is the unique normed* S_p*-series of order* -1 *with the polynomial*

$$G(X, Y) = a_m^{-(p+1)} F(a_m X, a_m Y).$$

This result shows that it suffices to determine all normed S_p-series of order -1 in order to obtain all S_p-series of arbitrary negative order.

15. The symmetric polynomial $F(X, Y)$ contains $P = [(p+1)(p+2)]/2$ essential coefficients, i.e., coefficients c_{rs} for which $0 \leqq r \leqq s \leqq p$. The most general S_p-series $f(z)$ of negative order depends therefore on P independent parameters c_{rs} in K and in addition on the negative integer m which is its order. On the other hand, the normed S_p-series $g[z]$ of order -1 involves only $P-1$ parameters for which we may take the elements of the set D defined in §12.

A further rather trivial reduction is possible. In the development

$$g[z] = z^{-1} + \sum_{h=0}^{\infty} b_h z^h$$

the constant term b_0 plays no essential role, and we may consider the S_p-series

$$h[z] = g[z] - a_0 = z^{-1} + \sum_{h=1}^{\infty} b_h z^h,$$

instead of $g[z]$. Also $h[z]$ is a normed S_p-series of order -1; we call it a *basic* S_p-series. Let

$$H(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{r=0}^{p} \sum_{s=0}^{p} E_{rs} X^r Y^s \qquad (E_{rs} = E_{sr})$$

be the polynomial of $h[z]$. By the proof in §12,

(26) $$E_{pp} = 0.$$

To this condition we can add the further one,

(27) $$E_{p-1,p} = E_{p,p-1} = 0.$$

It can easily be proved by evaluating the coefficient of the power

$$z^{-p^2-p+1}$$

in the identity

$$E(h[z^p], h[z]) = 0.$$

It follows that *the set of all basic S_p-series depends on only*

$$P - 2 = \frac{p(p+3)}{2}$$

independent parameters in K, viz., on the coefficients

$$E_{rs}, \quad \text{where} \quad 0 \leqq r \leqq s \leqq p,$$

but with the exclusion of the two trivial coefficients $E_{pp} = 0$ and $E_{p-1,p} = 0$.

We also see that, on allowing $C_0 \neq 0$ and C_1 to run over all constants, m to run over all negative integers, and $h[z]$ to run over all basic S_p-series, the expression

$$f(z) = C_0 h[z^{-m}] + C_1$$

describes all S_p-series of negative orders.

16. Theorem 3 showed that if $f(z)$ is an S_p-series of *positive* order m, then $f(z)^{-1}$ is an S_p-series of the negative order $-m$. We found that in this case the polynomial $F(X, Y)$ of $f(z)$ and the polynomial $F^0(X, Y)$ of $f(z)^{-1}$ had the forms (14a) and (14b) in §9, respectively. Both these polynomials contained only

$$Q = \frac{p(p+1)}{2}$$

essential coefficients on account of the conditions (13). If

$$f(z) = \sum_{h=m}^{\infty} a_h z^h,$$

the reciprocal function $f(z)^{-1}$ has a Laurent series which begins with the term $a_m^{-1} z^{-m}$. Hence

$$g(z) = a_m f(z)^{-1}$$

is a *normed* S_p-series, and even, by Theorem 3, a normed R_p-series; it has the negative order $-m$. If the polynomial is written in the form

$$G(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{r=0}^{p} \sum_{s=0}^{p} D_{rs} X^r Y^s \qquad (D_{rs} = D_{sr}),$$

then §9 and §12 imply that

$$D_{0s} = D_{s0} = 0 \quad \text{for} \quad s = 0, 1, \cdots, p, \quad \text{and} \quad D_{pp} = 0.$$

Here the first conditions hold because $g(z)$ is an R_p-series, and the last one since $g(z)$ is normed.

Conversely, by Theorem 4, these properties of the coefficients of $G(X, Y)$ show that *there exists to every negative integer* $-m$ *one and only one normed* S_p*-series* $g(z)$ *of the negative order* $-m$ *belonging to* $G(X, Y)$, *and this series is an* R_p*-series.*

Denote again by $g[z]$ the unique normed S_p-series of order -1 the polynomial of which is $G(X, Y)$. Then for every positive integer m the normed S_p-series $g[z^m]$ is the unique S_p-series of the negative order $-m$ belonging to $G(X, Y)$. In terms of this series the original S_p-series $f(z)$ can be written as

$$f(z) = a_m g[z^m]^{-1}. \tag{28}$$

It has in the present case no advantage to express in this formula $g[z]$ by the corresponding basic series $h[z]$.

We had put

$$Q = \frac{p(p+1)}{2}.$$

From the conditions for the coefficients D_{rs} of $G(X, Y)$, the normed series $g[z]$ depends only on the $Q-1$ essential coefficients

$$D_{rs}, \quad \text{where} \quad 1 \leqq r \leqq s \leqq p, \tag{29}$$

of $G(X, Y)$ where the trivial coefficient $D_{pp} = 0$ has been excluded. The $Q-1$ essential coefficients of $G(X, Y)$ may run independently over K. Therefore *the set of all* S_p*-series* $f(z)$ *of arbitrary positive orders depends on exactly* Q *parameters in* K *because* $a_m \neq 0$ *may still run over all constants.* Naturally $f(z)$ in addition depends on its positive order m.

Since $P > Q$, the following result is obtained.

THEOREM 6. *While the reciprocal of every* S_p*-series of positive order is again an* S_p*-series, but of negative order, the reciprocal of an* S_p*-series of negative order is not in general an* S_p*-series.*

17. There remains the study of the S_p-series of order 0. Let

$$f(z) = \sum_{h=0}^{\infty} a_h z^h, \quad \text{where} \quad a_0 \neq 0,$$

be such a series. Since $f(z)$ may not be a constant, there exists a positive suffix m such that

$$a_1 = a_2 = \cdots = a_{m-1} = 0, \quad \text{but} \quad a_m \neq 0.$$

Hence

$$f(z) = a_0 + g(z), \tag{30}$$

where

$$g(z) = \sum_{h=m}^{\infty} a_h z^h$$

is an S_p-series of the positive order m and so, by Theorem 3, is an R_p-series.

This implies, by §9, that the polynomial $G(X, Y)$ of $g(z)$ has the form

$$G(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p} \sum_{s=1}^{p} d_{rs} X^r Y^s \qquad (d_{rs} = d_{sr}).$$

Let similarly

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=0}^{p} \sum_{s=0}^{p} c_{rs} X^r Y^s \qquad (c_{rs} = c_{sr})$$

be the polynomial of $f(z)$. Then, by the relation (30),

$$F(X, Y) = G(X - a_0, Y - a_0) \tag{31}$$

identically in X and Y, a formula which defines the polynomial of every possible S_p-series of order 0. We see that such a series $f(z)$ depends on altogether $Q + 1$ parameters in K, namely the Q essential coefficients of $G(X, Y)$ and the coefficient $a_0 \neq 0$. In addition, $f(z)$ depends also on the arbitrary positive integer m, and it may be written in the form

$$f(z) = a_0 + a_m g[z^m]^{-1},$$

where $g[z]^{-1}$ denotes the unique normed S_p-series of order 1 with the polynomial $G(X, Y)$.

18. As has just been shown, the set of all S_p-series of order 0 depends on $Q + 1$ parameters in K. On the other hand, we proved in §16 that the set of all S_p-series of positive order involves only Q such parameters, and it follows from §§9–10 that the same is true for the set of all R_p-series.

Therefore, *in general*, an S_p-series $f(z)$ of order 0 is *not* an R_p-series and hence its reciprocal $f(z)^{-1}$ is *not* an S_p-series.

The question arises then whether there do exist exceptional S_p-series of order 0 which are R_p-series. This question can be answered as follows.

Let $f(z)$ be such an R_p-series of order 0, and let again $g(z)$ be the series of positive order defined by the equation (30); it is of positive order and therefore is an R_p-series. Let $G(X, Y)$ be the polynomial of $g(z)$ as given in the last section, and let similarly $F(X, Y)$ be the polynomial of $f(z)$. This polynomial has now the simpler form

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c_{rs} X^r Y^s \qquad (c_{rs} = c_{sr}).$$

The identity (31) can also be written as $F(X + a_0, Y + a_0) = G(X, Y)$ and is equivalent to

$$(32) \qquad \begin{aligned}(X + a_0)^{p+1} + (Y + a_0)^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c_{rs} (X + a_0)^r (Y + a_0)^s = \\ = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} d_{rs} X^r Y^s.\end{aligned}$$

On expanding the binomials on the left-hand side and comparing the coefficients of the different power products of X and Y, we find that the terms in X^{p+1} and Y^{p+1} cancel out, and that the terms in $X^r Y^s$, where r and s run from 1 to p, just allow to express the coefficients d_{rs} of $G(X, Y)$ as unique linear polynomials in the coefficients c_{rs}. The only remaining conditions still to be satisfied are that the coefficients of the powers $1, X, X^2, \cdots, X^p$ are equal to zero; by symmetry, this implies then the same for the coefficients of the powers $1, Y, Y^2, \cdots, Y^p$.

These $p + 1$ conditions can also be obtained by simply putting $Y = 0$ in (32). They are then contained in the single condition that

$$(33) \qquad (X + a_0)^{p+1} + a_0^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c_{rs} (X + a_0)^r a_0^s = X^{p+1}$$

identically in X.

If now the prime p is equal to 2, then on putting $X = -a_0$ we obtain the condition that $a_0^3 = (-a_0)^3$, hence that $a_0 = 0$, contrary to hypothesis. Thus the following special result holds.

THEOREM 7. *There are no R_2-series of order* 0. *In other words, if $f(z)$ is an S_2-series of order* 0, *then its reciprocal $f(z)^{-1}$ is not an S_2-series.*

A completely different position holds if the prime p is greater than 2.

Now, on replacing X by $X - a_0$ in (33), this identity takes the form

$$X^{p+1} + a_0^{p+1} + \sum_{r=1}^{p}\sum_{s=1}^{p} c_{rs} X^r a_0^s = \sum_{r=0}^{p+1} \binom{p+1}{r} X^r (-a_0)^{p-r+1}.$$

Here the terms in X^{p+1} and in a_0^{p+1} cancel out, and the remaining terms lead to the following system of p linear equations

$$(34) \qquad \sum_{s=1}^{p} c_{rs} a_o^s = \binom{p+1}{r} (-a_0)^{p-r+1} \qquad (r = 1, 2, \cdots, p)$$

for the coefficients c_{rs}. Each of the p diagonal coefficients c_{rr} occurs in one and only one of these equations and has the coefficient $a_0^r \neq 0$. These p equations are therefore non-contradictory and allow to express the c_{rr} linearly in terms of the remaining

$$Q - p = \frac{p(p-1)}{2}$$

essential coefficients

(35) $$c_{rs}, \quad \text{where} \quad 1 \leqq r < s \leqq p,$$

and of the coefficient a_0. Thus the following result has been established.

For $p \geqq 3$ the set of all R_p-series of order 0 *depends on $Q - p + 1$ parameters in K, viz., on the coefficients* (35) *and on a_0.*

Since both sides of the equation (34) have the common factor a_0, we further find that *if $F(X, Y)$ is given, a_0 has at most $p - 1$ possible values.* For otherwise these equations are satisfied identically in a_0, and hence

$$c_{r,p-r+1} = (-1)^{p-r+1}\binom{p+1}{r} \text{ for } r = 1, 2, \cdots, p, \text{ and otherwise } c_{rs} = 0.$$

Therefore

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p} (-1)^{p-r+1}\binom{p+1}{r} X^r Y^{p-r+1} = (X - Y)^{p+1}.$$

However, the only Laurent series $f(z)$ of order 0 satisfying the functional equation

$$F(f(z^p), f(z)) = (f(z^p) - f(z))^{p+1} = 0$$

is an arbitrary constant and so is not an S_p-series.

19. In the preceding sections we studied in detail on how many parameters does the general S_p-series $f(z)$ depend. The results were found to depend on whether $f(z)$ was of negative order, of positive order, or of order 0.

It was also found that every S_p-series could be expressed in terms of the normed S_p-series $g[z]$ of order -1, and when we were dealing with S_p-series of negative order, this normed series could be replaced by the corresponding basic series $h[z]$ in which the constant term was equal to zero.

In the next chapter we shall deal in detail with the relation between such a basic series $h[z]$ and its polynomial $H(X, Y)$.

Before studying this question, it is, however, appropriate to add some remarks on the S_p-series $f(z)$ for which the corresponding polynomial $F(X, Y)$ is reducible.

20. Let for the moment $f(z)$ be an S_p-series of arbitrary order, but with the property that *its polynomial* $F(X, Y)$ *is reducible*. By §7, $F(X, Y)$ allows then a factorisation

$$F(X, Y) = hH(X, Y)H(Y, X)$$

where $h \neq 0$ is a constant, and $H(X, Y)$ is an irreducible polynomial which has the exact degree 1 in X and p in Y, hence without loss of generality, may be assumed to have the form

$$H(X, Y) = -X + \sum_{s=0}^{p} \gamma_s Y^s$$

where $\gamma_0, \cdots, \gamma_{p-1}, \gamma_p \neq 0$ are certain constants. Hence

$$F(X, Y) = h\left(-Y + \sum_{r=0}^{p} \gamma_r X^r\right)\left(-X + \sum_{s=0}^{p} \gamma_s Y^s\right). \tag{36}$$

Since X^{p+1} is the term of $F(X, Y)$ in the highest power of S, it follows at once that

$$h\gamma_p = -1.$$

Let now, firstly, $f(z)$ be the *negative* order m, and let it be *normed*,

$$f(z) = z^m + \cdots.$$

The identity

$$H(f(z^p), f(z)) = 0,$$

which has already been used before, takes the explicit form

$$-(z^{mp} + \cdots) + \sum_{s=0}^{p} \gamma_s (z^m + \cdots)^s = 0.$$

Its term in z^{mp} has the coefficient $-1 + \gamma_p$ which must vanish; hence

$$\gamma_p = 1, \quad h = -1.$$

Hence the following result holds for reducible $F(X, Y)$.

If $f(z)$ is normed and of negative order, then its polynomial has the form

$$F(X, Y) = -\left(X^p - Y + \sum_{r=0}^{p-1} \gamma_r X^r\right)\left(Y^p - X + \sum_{s=0}^{p-1} \gamma_s Y^s\right), \tag{37}$$

hence depends on p parameters $\gamma_0, \gamma_1, \cdots, \gamma_{p-1}$ in K.

We must add $a_m \neq 0$ as a further parameter if $f(z)$ is not normed. A similar proof shows that *if $f(z)$ is basic, then also $\gamma_{p-1} = 0$.*

21. The two remaining cases lead to rather trivial results, as follows.

Let, secondly, $f(z)$ be of positive order and normed. Then, on one hand, the reducible polynomial $F(X, Y)$ has the form (36). On the other hand, it follows from §9 that $F(X, Y)$ has the special form

$$F(X, Y) = X^{p+1} + Y^{p+1} + \sum_{r=1}^{p} \sum_{s=1}^{p} c_{rs} X^r Y^s, \tag{14a}$$

and here, by §§11–12,

$$c_{pp} = -1.$$

On comparing these two expressions for $F(X, Y)$, it follows immediately that

$$\gamma_0 = \gamma_1 = \cdots = \gamma_{p-1} = 0, \quad \gamma_p = 1, \quad h = -1$$

and that therefore

$$F(X, Y) = -(X^p - Y)(Y^p - X).$$

The corresponding equation

$$H(f(z^p), f(z)) = f(z^p) - f(z)^p = 0$$

is easily seen to have no other normed solution of the positive order m than the monomial $f(z) = z^m$. The general S_p-series of positive order with a reducible polynomial $F(X, Y)$ has the form $f(z) = a_m z^m$ where again $a_m \neq 0$ is a constant.

Thirdly, let $f(z)$ be of order 0 and not necessarily normed. Since $f(z) - a_0$ is of positive order and still has a reducible polynomial, it follows now that $f(z)$ is a polynomial

$$f(z) = a_0 + a_m z^m, \quad \text{where} \quad a_0 \neq 0 \text{ and } a_m \neq 0$$

are arbitrary constants and m is any positive integer. The polynomial $F(X, Y)$ of $f(z)$ has then the special form

$$F(X, Y) = -((X - a_0)^p - (Y - a_0))((Y - a_0)^p - (X - a_0)).$$

In future an S_p-series $f(z)$ will be called *general* or *special*, according as to whether its polynomial $F(X, Y)$ is *irreducible* or *reducible*, respectively. By what has just been proved, special S_p-series of non-negative orders are monomial or binomial polynomials and so have little interest. On the other hand, special S_p-series of negative orders have non-trivial properties.

Chapter 2. **Basic series and their polynomials**

22. As was found in the first chapter, S_p-series of negative orders are more general than those of non-negative orders, and the latter can always be expressed in terms of S_p-series of negative orders.

It is for this reason that we shall from now on consider only S_p-series of negative orders. It will, in fact, be sufficient to deal only with *basic series*, i.e., with normed series of order -1 in which the constant term is missing.

Let

$$h[z] = z^{-1} + \sum_{h=1}^{\infty} b_h z^h \tag{38}$$

be such a basic S_p-series, and let, as in §15, its polynomial $H(X, Y)$ be written in the form

$$H(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{r=0}^{p} \sum_{s=0}^{p} E_{rs} X^r Y^s \qquad (E_{rs} = E_{sr}). \tag{39}$$

As was then already mentioned, the coefficients of $H(X, Y)$ satisfy the conditions

$$E_{pp} = E_{p-1,p} = E_{p,p-1} = 0. \tag{40}$$

We note once more that every S_p-series of negative order m can be expressed in the form

$$f(z) = a_0 + a_m h[z^{-m}]$$

where a_0 and $a_m \neq 0$ are arbitrary constants, and that then, by §8, the polynomial $F(X, Y)$ of $f(z)$ is given by

$$a_m^{p+1} F(X, Y) = H(a_0 + a_m X, a_0 + a_m Y).$$

23. Theorem 4 and its corollary imply that, when $H(X, Y)$ is given, the basic S_p-series $h[z]$ is unique, and all its coefficients b_h can be expressed as polynomials in the coefficients E_{rs} of $H(X, Y)$. It was then already mentioned that, conversely, it is similarly possible to write the coefficients H_{rs} as polynomials in a certain finite number of the coefficients b_h. This representation will now be established, but explicitly only in the lowest two cases when $p = 2$ and $p = 3$ because for larger primes p the formulae become rather complicated.

The idea, on which this calculation is based, is, however, quite simple. It is founded on the definition of $H(X, h[z])$ as the monic polynomial in X of degree $p+1$ which has as its zeros the $p+1$ elements

$$h[z^p], h[z^{1/p}], h[\varepsilon z^{1/p}], \cdots, h[\varepsilon^{p-1} z^{1/p}]$$

of the set Σ_h. It is thus required to evaluate the $p+1$ elementary symmetric functions, $s_1, s_2, \cdots, s_{p+1}$ say, of these $p+1$ series, and then

$$H(X, h[z]) = X^{p+1} - s_1 X^p + s_2 X^{p-1} - + \cdots \pm s_{p+1}.$$

Here the main problem lies in expressing the s_j as polynomials in $h[z]$ and in the Laurent coefficients b_h.

24. To begin with, the definition (38) of $h[z]$ is easily seen to imply an infinite sequence of identities

$$\begin{aligned} z^{-1} &= h[z] + \cdots, \\ z^{-2} &= h[z]^2 - 2b_1 + \cdots, \\ z^{-3} &= h[z]^3 - 3b_1 h[z] - 3b_2 + \cdots, \\ z^{-4} &= h[z]^4 - 4b_1 h[z]^2 - 4b_2 h[z] + (2b_1^2 - 4b_3) + \cdots, \end{aligned} \tag{41}$$

etc., when the dots denote formal power series in negative powers of $h[z]$. Further positive powers of z are expressible as formal power series in such negative powers.

Consider now, firstly, the simplest case when $p = 2$. The set Σ_h consists of the three series

$$h[z^2] = z^{-2} + \sum_{h=1}^{\infty} b_h z^{2h},$$

$$h[z^{1/2}] = z^{1/2} + \sum_{h=1}^{\infty} b_h z^{h/2}, \quad h[-z^{1/2}] = -z^{-1/2} + \sum_{h=1}^{\infty} (-1)^h z^{h/2}.$$

Here

$$h[z^{1/2}] + h[-z^{1/2}] = 2\sum_{h=1}^{\infty} b_{2h} z^h \quad \text{and} \quad h[z^{1/2}]h[-z^{1/2}] = -z^{-1} - \sum_{h=0}^{\infty} \beta_h z^h,$$

where the new coefficients β_h are defined by

$$\beta_0 = 2b_1, \quad \beta_1 = 2b_3 + b_1^2, \quad \text{and}$$

$$\beta_h = 2b_{2h+1} + 2\sum_{j=1}^{h-1} (-1)^{j-1} b_j b_{2h-j} + (-1)^{h-1} b_h^2 \quad \text{for} \quad h \geqq 2.$$

The three elementary symmetric functions

$$\begin{aligned} s_1 &= h[z^2] + h[z^{1/2}] + h[-z^{1/2}], \\ s_2 &= h[z^2]h[z^{1/2}] + h[z^2]h[-z^{1/2}] + h[z^{1/2}]h[-z^{1/2}], \\ s_3 &= h[z^2]h[z^{1/2}]h[-z^{1/2}] \end{aligned}$$

can therefore be written as the Laurent series

$$s_1 = z^{-2} - 2b_1 + \cdots,$$

$$(42) \qquad s_2 = (2b_2 - 1)z^{-1} + (2b_4 - 2b_1) + \cdots,$$

$$s_3 = -z^{-3} - 2b_1 z^{-2} - (2b_3 + b_1^2)z^{-1} - (2b_5 + 2b_1 b_3 - b_2^2) + \cdots,$$

where the dots now denote power series in positive powers of z.

Replace now in (42) the terms in non-positive powers of z by their expressions (41); they become then Laurent series in $h[z]$. But we know that s_1, s_2, and s_3 are polynomials in $h[z]$; hence the terms in negative powers of $h[z]$ must cancel out. The explicit expressions of s_1, s_2, and s_3 take therefore the form

$$s_1 = h[z]^2 - 2b_1,$$

$$s_2 = (2b_2 - 1)h[z] + (2b_4 - 2b_1),$$

$$s_3 = -h[z]^3 - 2b_1 h[z]^2 - (2b_3 + b_1^2 - 3b_1)h[z] - (2b_5 + 2b_1 b_3 - b_2^2 - 3b_2 - 4b_1^2).$$

From these expressions, it follows immediately that the polynomial $H(X, Y)$ of $h[z]$ has the following explicit form,

$$(43) \qquad \begin{aligned} H(X, Y) = {} & -(X^2 - Y)(Y^2 - X) + 2b_1(X^2 + Y^2) + 2b_2 XY + (2b_4 - 2b_1)X + \\ & + (2b_3 + b_1^2 - 3b_1)Y + (2b_5 + 2b_1 b_3 - b_2^2 - 3b_2 - 4b_1^2). \end{aligned}$$

By Theorem 2, $H(X, Y)$ is symmetric in X and Y. Hence

$$(44) \qquad 2b_4 = 2b_3 + b_1^2 - b_1.$$

The essential coefficients H_{rs} of $H(X, Y)$ are given by

$$H_{02} = 2b_1, \quad H_{11} = 2b_2,$$

$$H_{01} = 2b_3 + b_1^2 - 3b_1 = 2b_4 - 2b_1, \quad H_{00} = 2b_5 + 2b_1 b_3 - b_2^2 - 3b_2 - 4b_1^2,$$

and, conversely,

$$2b_1 = H_{02}, \quad 2b_2 = H_{11},$$

$$8b_3 = 4H_{01} - H_{02}^2 + 6H_{02}, \quad 2b_4 = H_{01} + H_{02},$$

$$16b_5 = 8H_{00} - 4H_{02}H_{01} + H_{02}^3 + 2H_{02}^2 - 2H_{11}^2 + 6H_{11}.$$

Thus for basic S_2-series $h[z]$ the polynomial $H(X, Y)$ is known if the first five coefficients

$$(45) \qquad b_1, b_2, b_3, b_4, b_5$$

are given where, however, these five coefficients have to satisfy the condition (44). Theorem 4 shows that no further restrictions on the coefficients (45) need be imposed. Thus, when the equation (44) holds, then there always exists exactly one basic S_2-series with the first five coefficients (45).

It should now be possible to express all further coefficients b_h $(h \geqq 6)$ of $h[z]$ in terms of the coefficients (45). As we shall in fact immediately prove, these coefficients b_h with $h \geqq 6$ can be written as polynomials in the coefficients (45), where the numerical coefficients lie in the prime field of K.

25. For this purpose, a set of recursive formulae will now be established which allows to evaluate step by step the successive coefficients b_h once the coefficients (45) are given. *Thus these recursive formulae depend only implicitly on the polynomial $H(X, Y)$ of $h[z]$.*

The construction is based on the two identities

$$s_1 = h[z^2] + h[z^{1/2}] + h[-z^{1/2}] = h[z]^2 - 2b_1,$$

$$s_2 = h[z^2](h[z^{1/2}] + h(-z^{1/2}]) + h[z^{1/2}]h[-z^{1/2}] = (2b_1 - 1)h[z] + (2b_4 - 2b_1)$$

of the last section. If in these identities

$$h[z], h[z^2], h[z^{1/2}], h[-z^{1/2}]$$

are replaced by their Laurent series, the resulting identities contain only integral powers of z. On comparing now the coefficients of these different powers and putting their sums equal to 0, the required recursive formulae are obtained. By means of a suitable linear combination, they can then be put in the following forms where k runs over all positive integers, and empty sums are to mean 0.

(46a)
$$b_{4k} = b_{2k+1} + \sum_{j=1}^{k-1} b_j b_{2k-j} + (b_k^2 - b_k)/2,$$

(46b)
$$b_{4k+1} = b_{2k+3} + \sum_{j=1}^{k} b_j b_{2k-j+2} - \sum_{j=1}^{2k-1} (-1)^{j-1} b_j b_{4k-j} + \\ + \sum_{j=1}^{k-1} b_j b_{4k-4j} - b_2 b_{2k} + (b_{k+1}^2 - b_{k+1})/2 + (b_{2k}^2 + b_{2k})/2,$$

(46c)
$$b_{4k+2} = b_{2k+2} + \sum_{j=1}^{k} b_j b_{2k-j+1},$$

(46d)
$$b_{4k+3} = b_{2k+4} + \sum_{j=1}^{k+1} b_j b_{2k-j+3} - \sum_{j=1}^{2k} (-1)^{j-1} b_j b_{4k-j+2} + \\ + \sum_{j=1}^{k} b_j b_{4k-4j+2} - b_2 b_{2k+1} - (b_{2k+1}^2 - b_{2k+1})/2.$$

There is a third identity

$$s_3 = h[z^2]h[z^{1/2}]h[-z^{1/2}] =$$

$$= -(h[z]^3 + 2b_1 h[z]^2 + (2b_3 + b_1^2 - 3b_1)h[z] + (2b_5 + 2b_1 b_3 - b_2^2 - 3b_2 - 4b_1^2)),$$

to which the same method may be applied. This leads then to a further set of recursive formulae. But while the formulae (46) involve at most products of two coefficients b_h, and in addition have rather simple laws, the new formulae contain products of up to three factors b_h and are much more complicated. The first three of the new relations are as follows.

$$2b_7 + 2b_1b_5 - 2b_2b_4 + b_3^2 =$$
$$= b_1(3b_3 + 3b_1^2) + 4b_1b_2 + (2b_3 + b_1^2 - 3b_1)b_1,$$

$$2b_9 + 2b_1b_7 - 2b_2b_6 + 2b_3b_5 - b_4^2 =$$
$$= 2b_1^2 + (4b_4 + 4b_1b_2) + 2b_1(2b_3 + b_1^2) + (2b_3 + b_1^2 - 3b_1)b_2,$$

$$2b_{11} + 2b_1b_9 - 2b_2b_8 + 2b_3b_7 - 2b_4b_6 + b_5^2 =$$
$$= b_1(2b_3 + b_1^2) + b_2 + (3b_5 + 6b_1b_3 + 3b_2^2 + b_1^3) +$$
$$+ 2b_1(2b_4 + 2b_1b_2) + (2b_3 + b_1^2 - 3b_1)b_3,$$

and the further ones get even more involved. We can fortunately disregard all these formulae because, by Theorem 4, they are necessarily consequences of (44) and of the recursive formulae (46).

26. Next let $p = 3$, and let

$$h[z] = z^{-1} + \sum_{h=1}^{\infty} b_h z^h$$

be any basic S_3-series. The corresponding set Σ_h consists of the four series

$$h[z^3] = z^{-3} + \sum_{h=1}^{\infty} b_h z^{3h},$$

$$\text{and} \quad h[\varepsilon^j z^{1/3}] = \varepsilon^{-j} z^{-1/3} + \sum_{h=1}^{\infty} b_h \varepsilon^{hj} z^{h/3} \quad \text{where} \quad j = 0, 1, 2.$$

Here ε denotes a primitive third root of unity in K.

The problem is again to express the elementary symmetric functions s_1, s_2, s_3, s_4 of the elements of Σ_h as polynomials in $h[z]$. This can be done just as for $p = 2$, but the calculations become now more complicated. On substituting the expressions so obtained for the s_j in $H(X, Y)$, this polynomial assumes the following explicit form:

$$
\begin{aligned}
H(X, Y) = \\
&= -(X^3 - Y)(Y^3 - X) + 3b_1(X^3 Y + XY^3) + 3b_2(X^3 + Y^3) + 3b_3 X^2 Y^2 + \\
&+ (3b_4 + 3b_1 b_2)X^2 Y + 3b_6 XY^2 + (3b_5 - 3b_1 b_3 + 3b_2^2 + b_1^3 - 4b_1)X^2 + \\
&+ (3b_9 - 6b_1 b_3 - 3b_1)Y^2 + (3b_7 + 3b_1 b_5 + 3b_2 b_4 - 3b_3^2 - 9b_1^2)XY + \\
&+ (3b_8 - 3b_1 b_6 + 6b_2 b_5 - 3b_3 b_4 + 3b_1^2 b_4 - 3b_1 b_2 b_3 + b_2^3 - 9b_1 b_2 - 4b_2)X + \\
&+ (3b_{10} + 3b_1 b_8 + 3b_2 b_7 - 6b_3 b_6 + 3b_4 b_5 - 6b_1 b_4 - 6b_1^2 b_2 - 9b_1 b_2 - 3b_2)Y + \\
&+ (3b_{11} - 3b_1 b_9 + 6b_2 b_8 - 3b_3 b_7 + 3b_1^2 b_7 - 3b_4 b_6 - 3b_1 b_2 b_6 + 3b_5^2 + \\
&+ 3b_2^2 b_5 - 6b_1 b_5 - 3b_1 b_3 b_5 + 3b_1 b_4^2 - 3b_2 b_3 b_4 + b_3^3 + 6b_1^2 b_3 - \\
&- 4b_3 - 6b_1 b_2^2 - 9b_2^2 - 2b_1^4 + 2b_1^2 + b_1).
\end{aligned} \tag{47}
$$

Since $H(X, Y)$ is symmetric in X and Y, the Laurent coefficients b_h that occur in this representation must satisfy the following three symmetry conditions:

$$
\begin{aligned}
b_6 &= b_4 + b_1 b_2, \\
b_9 &= b_5 + b_1 b_3 + b_2^2 + (b_1^3 - b_1)/3, \\
b_{10} &= -b_1 b_8 + b_8 - b_2 b_7 + 2b_3 b_6 - b_1 b_6 - b_4 b_5 + 2b_2 b_5 - b_3 b_4 + \\
&\quad + b_1^2 b_4 + 2b_1 b_4 - b_1 b_2 b_3 + 2b_1^2 b_2 + (b_2^3 - b_2)/3.
\end{aligned} \tag{48}
$$

Thus for $p = 3$ the eight Laurent coefficients

$$b_1, b_2, b_3, b_4, b_5, b_7, b_8, b_{11} \tag{49}$$

of $h[z]$ determine b_6, b_9, and b_{10} and hence also determine the polynomial $H(X, Y)$.

The formulae (47) and (48) allow to write the eight essential coefficients

$$E_{13}, E_{03}, E_{22}, E_{12}, E_{02}, E_{11}, E_{01}, E_{00} \tag{50}$$

of $H(X, Y)$ as polynomials in the eight Laurent coefficients (49) of $h[z]$, and conversely, the latter can be expressed as polynomials in the coefficients (50) of $H(X, Y)$.

27. Just as for $p = 2$, so also in the present case $p = 3$ we can establish recursive formulae for the coefficients b_h of $h[z]$. It is to be expected that these formulae will be rather more complicated. They again involve the coefficients of $H(X, Y)$ only implicitly.

Put

$$k(z)=\sum_{h=1}^{\infty} b_h z^{3h};$$

$$k_0(z)=\sum_{h=1}^{\infty} b_{3h}z^h,\quad k_1(z)=\sum_{h=0}^{\infty} b_{3h+1}z^h,\quad k_2(z)=\sum_{h=0}^{\infty} b_{3h+2}z^h.$$

The four elements of Σ_h can then be written in the form

$$h[z^3]=z^{-3}+k(z);$$

$$h[\varepsilon^j z^{1/3}]=\varepsilon^{-j}z^{-1/3}+k_0(z)+\varepsilon^j z^{1/3}k_1(z)+\varepsilon^{2j}z^{2/3}k_2(z)\quad \text{for}\quad j=0,1,2.$$

On evaluating their elementary symmetric functions, the terms in fractional powers of z cancel out, and the s_j become polynomials in z, $k(z)$, $k_0(z)$, $k_1(z)$, and $k_2(z)$. On the other hand, the expression (47) for $H(X,Y)$ allows to write them also as polynomials in $h[z]$ and in the first eleven coefficients b_h of $h[z]$.

On carrying out these trivial calculations, we obtain the following set of formulae.

$$s_1=h[z]^3-3b_1h[z]-3b_2=z^{-3}+k(z)-3k_0(z),$$

$$s_2=3b_3h[z]^2+3b_6h[z]+(3b_9-6b_1b_3-3b_1)=$$

$$=3z^{-3}k_0(z)+3k(z)k_0(z)+3k_0(z)^2-3k_1(z)-3zk_1(z)k_2(z)$$

$$s_3=-3b_1h[z]^3-3b_6h[z]^2+(1-3b_7-3b_1b_5-3b_2b_4+3b_3^2+9b_1^2)h[z]-$$

$$-(3b_8-3b_1b_6+6b_2b_5-3b_3b_4+3b_1^2b_4-3b_1b_2b_3+b_2^3-9b_1b_2-4b_2)=$$

$$=3z^{-3}(k_0(z)^2-k_1(z)-zk_1(z)k_2(z))+z^{-1}+k_0(z)^3+zk_1(z)^3+$$

$$+\ 3k_2(z)+3zk_2(z)^2+z^2k_2(z)^3-3k_0(z)k_1(z)-3zk_0(z)k_1(z)k_2(z)+$$

$$+\ 3k(z)(k_0(z)^2-k_1(z)-zk_1(z)k_2(z)).$$

Here replace again $h[z]$, $k(z)$, $k_0(z)$, $k_1(z)$, and $k_2(z)$, by their series and then compare on both sides of the resulting identities the coefficients of the different powers of z: For the lower powers of z this leads only to trivial identities, or to the symmetry conditions (48), or to equations that follow from these conditions.

But for $h\geqq 4$ the wanted recursive formulae are obtained. They have the following forms:

$$\text{(51a)}\qquad b_{3h}=b_{h+2}-b_1b_h-\frac{1}{3}b_{h/3}+\sum_{j=1}^{h} b_jb_{h-j+1}+\frac{1}{3}\sum_{h_1+h_2+h_3=h} b_{h_1}b_{h_2}b_{h_3},$$

where $b_{h/3}$ denotes 0 if h is not divisible by 3, and in the multiple sum h_1, h_2, h_3 denote positive integers of the sum h;

$$b_{3h+1} = -b_6b_h - 2b_3b_{h+1} - b_1b_{h+1} - \frac{1}{3}b_{(h+3)/3} + b_{h+5} - b_3\sum_{j=1}^{h-1} b_jb_{h-j} +$$

(51b)
$$+\sum_{j=1}^{h+3} b_jb_{h-j+4} + \sum_{j=1}^{h-1} b_{3j}b_{3h-3j} - \sum_{j=0}^{h-1} b_{3j+1}b_{3h-3j-1} +$$

$$+\sum_{1\leqq j<h/3} b_jb_{3h-9j} + \frac{1}{3}\sum_{h_1+h_2+h_3=h+3} b_{h_1}b_{h_2}b_{h_3},$$

where again $b_{(h+3)/3}$ denotes 0 if h is not divisible by 3, and where in the last sum h_1, h_2, h_3 run over all positive integers of the sum $h+3$;

$$b_{3h+2} = \frac{1}{3}(1 - 3b_7 - 3b_1b_5 - 3b_2b_4 + 3b_3^2 + 9b_1^2)b_h - 2b_6b_{h+2} - 3b_1b_{h+2} -$$

$$- b_6b_{h+3} - 2b_3b_{h+4} - b_{(h+6)/3} - b_1b_{h+6} + b_{h+8} - b_6\sum_{j=1}^{h-1} b_jb_{h-j} -$$

$$- 3b_1\sum_{j=1}^{h} b_jb_{h-j+1} - b_3\sum_{j=1}^{h+2} b_jb_{h-j+3} + \sum_{j=1}^{h+6} b_jb_{h-j+7} +$$

$$+\sum_{1\leqq j\leqq h/3} b_jb_{3h-9j+1} + \sum_{1\leqq j<(h+3)/3} b_jb_{3h-9j+9} + \sum_{j=1}^{h} b_{3j}b_{3h-3j+1} -$$

(51c)
$$-\sum_{j=0}^{h-1} b_{3j+2}b_{3h-3j-1} - b_1\sum_{h_1+h_2+h_3=h} b_{h_1}b_{h_2}b_{h_3} +$$

$$+\frac{1}{3}\sum_{h_1+h_2+h_3=h+6} b_{h_1}b_{h_2}b_{h_3} - \sum_{3h_1+h_2+h_3=h} b_{h_1}b_{3h_2}b_{3h_3} +$$

$$+\sum_{3h_1+k_2+k_3=h-1} b_{h_1}b_{3k_2+1}b_{3k_3+2} - \frac{1}{3}\sum_{h_1+h_2+h_3=h} b_{3h_1}b_{3h_2}b_{3h_3} +$$

$$+\sum_{h_1+k_2+k_3=h-1} b_{3h_1}b_{3k_2+1}b_{3k_3+2} - \frac{1}{3}\sum_{k_1+k_2+k_3=h-1} b_{3k_1+1}b_{3k_2+1}b_{3k_3+1} -$$

$$-\frac{1}{3}\sum_{k_1+k_2+k_3=h-2} b_{3k_1+2}b_{3k_2+2}b_{3k_3+2},$$

where also $b_{(h+6)/3}$ denotes 0 if h is not divisible by 3, and where in the multiple sums letters h_i run over positive integers and letters k_i over non-negative integers.

There is a further set of recursive formulae corresponding to the two expressions for the symmetric function s_4, but I have not tried to determine this set since it is naturally a consequence of the three sets (51).

28. For $p \geqq 5$ I have not tried to determine $H(X, Y)$ and the full sets of recursive formulae analogous to (46) for $p=2$ and (51) for $p=3$; for they become very complicated.

We note, however, that the first symmetric function

$$s_1 = h[z^p] + \sum_{j=0}^{p-1} h[\varepsilon^j z^{1/p}],$$

where ε denotes again a primitive pth root of unity in K, may still easily be written as a polynomial in $h[z]$, so that at least for the special coefficients b_{hp} a set of recursive formulae can be deduced. This is done as follows.

In the formula (41) of §24, we had for $1 \leqq n \leqq 4$ written the expressions for the terms in non-negative powers of $h[z]$ of the developments of z^{-n} into Laurent series in descending powers of $h[z]$. For $n=5$ and $n=7$ one similarly finds that

$$(52) \quad z^{-5} = h[z]^5 - 5b_1 h[z]^3 - 5b_2 h[z]^2 + 5(b_1^2 - b_3)h[z] + 5(b_1 b_2 - b_4) + \cdots$$

and

$$(53) \quad \begin{aligned} z^{-7} = {} & h[z]^7 - 7b_1 h[z]^5 - 7b_2 h[z]^4 + 7(2b_1^2 - b_3)h[z]^3 + 7(3b_1 b_2 - b_4)h[z]^2 + \\ & + 7(2b_1 b_3 + b_2^2 - b_2^3 - b_5)h[z] + 7(b_1 b_4 + b_2 b_3 - b_1^2 b_2 - b_6) + \cdots \end{aligned}$$

From the definition of s_1 as the first elementary symmetric function of the elements of Σ_h it follows, on the other hand, that

$$(54) \qquad s_1 = z^{-p} + \sum_{h=1}^{\infty} b_h z^{hp} + p \sum_{h=1}^{\infty} b_{hp} z^h.$$

Replace now again $h[z]$ in (52) and (53) by its Laurent series and compare the coefficients of the different powers z^h, where $h \geqq 1$, in the resulting series with those in the series (54). We obtain then the required recursive formulae for b_{hp} in the two cases when $p=5$ or $p=7$.

We shorten these formulae by adopting the following notations. Similarly as before, put $b_{h/p}$ equal to 0 if h is not divisible by p. Further, for any two positive integers m and n put

$$\sigma(m\,;n) = \sum b_{h_1} b_{h_2} \cdots b_{h_m}$$

where the summation is extended over all sets of m positive indices $h_1, h_2, \cdots, h_m$ which satisfy the equation

$$h_1 + h_2 + \cdots + h_m = n,$$

but put the sum equal to 0 if it is empty.

For $p=5$ the resulting formula is still reasonably simple, viz.

$$b_{5h} = b_{h+4} - 3b_1 b_{h+2} - 2b_2 b_{h+1} + (b_1^2 - b_3)b_h + 2\sigma(2; h+3) -$$

(55)
$$- 3b_1\sigma(2; h+1) - b_2\sigma(2; h) + 2\sigma(3; h+2) - b_1\sigma(3; h) +$$

$$+ \sigma(4; h+1) + \frac{1}{5}(\sigma(5; h) - b_{h/5}).$$

It becomes the trivial identity for $h = 1$, but for $2 \leqq h \leqq 5$ is equivalent to

$$b_{10} = b_6 + b_1 b_4 + b_2 b_3,$$

$$b_{15} = b_7 + b_1 b_5 + 2b_2 b_4 + b_3^2 + b_1^2 b_3 + b_1 b_2^2,$$

$$b_{20} = b_8 + b_1 b_6 + 2b_2 b_5 + 3b_3 b_4 + b_2^3 + 4b_1 b_2 b_3 + b_1^2 b_4 + b_1^3 b_3,$$

$$b_{25} = b_9 + b_1 b_7 + 2b_2 b_6 + b_1^2 b_5 + 3b_3 b_5 + 2b_4^2 - 2b_1 b_2 b_4 + 3b_2 b_3 + b_1^3 b_3 +$$

$$+ 3b_1^2 b_2^2 + \frac{1}{5}(b_1^5 - b_1) - b_2^3.$$

Here the equations for b_{10} and b_{15} occurred already amongst the recursive formulae for $p = 2$, and for $p = 3$, respectively.

For the prime $p = 7$ the expression for b_{7h} is far more complicated.

$$b_{7h} = b_{h+6} - 5b_1 b_{h+4} - 4b_2 b_{h+3} + 3(2b_1^2 - b_3)b_{h+2} + 2(3b_1 b_2 - b_4)b_{h+1} +$$

$$(2b_1 b_3 + b_2^2 - b_1^3 - b_5)b_h + 3\sigma(2; h+5) - 10b_1\sigma(2; h+3) -$$

$$- 6b_2\sigma(2; h+2) + 3(2b_1^2 - b_3)\sigma(2; h+1) +$$

(56)
$$+ (3b_1 b_2 - b_4)\sigma(2; h) + 5\sigma(3; h+4) - 10b_1\sigma(3; h+2) -$$

$$- 4b_2\sigma(3; h+1) + (2b_1^2 - b_3)\sigma(3; h) + 5\sigma(4; h+3) -$$

$$- 5b_1\sigma(4; h+1) - b_2\sigma(4; h) + 3\sigma(5 h+2) - b_1\sigma(5; h) +$$

$$+ \sigma(6; h+1) + \frac{1}{7}(\sigma(7; h) - b_{h/7}).$$

For $h = 1$ this formula becomes again the trivial identity, but for $h = 2$ and $h = 3$ it becomes

$$b_{14} = b_8 + b_1 b_6 + b_2 b_5 + b_3 b_4,$$

$$b_{21} = b_9 + b_1 b_7 + 2b_2 b_6 + 2b_3 b_5 + b_1^2 b_5 + b_4^2 + 2b_1 b_2 b_4 + b_1 b_3^2 + b_2^2 b_3.$$

These two equations occurred already amongst the recursive formulae for $p = 2$, and for $p = 3$, respectively.

29. As we mentioned already, the explicit determination of $H(X, Y)$ for $p \geqq 5$ in terms of the Laurent coefficients b_h of $h[z]$ becomes excessively complicated. But it is still possible to formulate some general laws about this interdependence.

By the corollary to Theorem 4, each of the Laurent coefficients b_h can be expressed as a polynomial over K in the coefficients E_{rs} of $H(X, Y)$. Conversely, the formulae (43) and (44), and (47) and (48), show that for $p = 2$ the E_{rs} are polynomials in the four Laurent coefficients

$$b_1, b_2, b_3, \quad \text{and} \quad b_5,$$

and for $p = 3$ in the eight Laurent coefficients

$$b_1, b_2, b_3, b_4, b_5, b_7, b_8, \quad \text{and} \quad b_{11}.$$

In each of these two cases a multiple of the highest coefficient ($2b_5$ or $3b_{11}$) occurs in the constant term E_{00} of $H(X, Y)$.

An analogous result is still valid for $p \geqq 5$. Also here the highest occurring Laurent coefficient, b_{h^*} say, arises in the constant term E_{00} of $H(X, Y)$, and its suffix h^* can be found by the following simple consideration.

The highest elementary symmetric function s_{p+1} of the elements of Σ_h is the product

$$s_{p+1} = \left(z^{-p} + \sum_{h=1}^{\infty} b_h z^{hp} \right) \prod_{j=0}^{p-1} \left(\varepsilon^{-j} z^{-1/p} + \sum_{h=1}^{\infty} b_h \varepsilon^{jh} z^{h/p} \right).$$

Since p is odd, it can also be written as

$$s_{p+1} = h[z]^{p+1} + \sum_{s=0}^{p} E_{0s} h[z]^s.$$

On replacing in the second formula $h[z]$ by its Laurent series, two different developments of s_{p+1} into Laurent series are obtained, and on comparing them, it is obvious that the coefficient of z^0 will form a part of E_{00} and at the same time will involve the Laurent coefficient b_{h^*}. The product formula for s_{p+1} shows easily that the coefficient of z^0 involves a term

$$pb_{p^2+p-1},$$

but none with a factor b_h where $h \geqq p^2 + p$. Hence $h^* = p^2 + p - 1$. It is also clear that b_{h^*} cannot occur in any one of the other elementary symmetric functions $s_1, s_2, \cdots, s_p$.

Thus $H(X, Y)$ is already determined if all the Laurent coefficients

$$(57) \qquad b_h, \quad \textit{where} \quad 1 \leqq h \leqq p^2 + p - 1,$$

are given.

On the other hand, $H(X, Y)$ depends by

(40) $$E_{pp} = E_{p-1,p} = E_{p,p-1} = 0$$

on exactly

$$P - 2 = \frac{p(p+3)}{2} - 1$$

essential coefficients E_{rs}, and the $p^2 + p - 1$ Laurent coefficients (57) can be expressed as polynomials in the latter.

This implies that *the coefficients* (57) *have still to satisfy*

(58) $$(p^2 + p - 1) - \left(\frac{p(p+3)}{2} - 1\right) = \frac{p(p-1)}{2}$$

independent algebraic conditions. Denote by C_p this set of $p(p-1)/2$ conditions.

For $p = 2$,

$$p^2 + p - 1 = 5, \qquad \frac{p(p-1)}{2} = 1,$$

and for $p = 3$,

$$p^2 + p - 1 = 11, \qquad \frac{p(p-1)}{2} = 3,$$

in agreement with the results in §24 and §26.

The proof just given does not supply any information about the *form* of the equations in C_p. Presumably these equations will again be equivalent to the symmetry conditions

$$E_{rs} = E_{sr} \qquad (r, s = 0, 1, \cdots, p).$$

It seems further plausible that in each of the equations of C_p the coefficient b_h of largest suffix will occur linearly, just as it was for $p = 2$ and $p = 3$.

The Laurent coefficients

(59) $$b_h, \qquad \text{where} \quad h \geqq p^2 + p,$$

do *not* occur in the explicit expression for $H(X, Y)$, but, by the corollary to Theorem 4, they can be expressed as polynomials in the coefficients E_{rs} of $H(X, Y)$. The latter, on the other hand, can be written as polynomials in the $p^2 + p - 1$ Laurent coefficients (57). *Hence the coefficients* (59) *can all be expressed as polynomials in the coefficients* (57). This suggests that they are already polynomials in only certain

$$\frac{p(p+3)}{2} - 1$$

of the coefficients (57). This assertion certainly holds in the two lowest cases $p=2$ and $p=3$, as is obvious from the form of the equations (44), and (48), respectively.

Chapter 3. **Analytic S_p-functions**

30. So far, only formal S_p-series over a field K have been considered. Here the only restrictions on K were that its characteristic was distinct from p and that it contained a primitive pth root of unity.

Now specialise K and assume that it possesses a non-trivial valuation $w(x)$. The question arises then whether a given S_p-series over K has a region of convergence relative to the metric induced by the valuation if the indeterminate z is replaced by a variable in K.

We shall restrict the discussion of this problem to the particularly interesting case when K is the complex number field $\underset{\sim}{C}$, and $w(x)$ is the absolute value $|x|$ in $\underset{\sim}{C}$.

A given S_p-series

$$f(z)=\sum_{h=m}^{\infty} a_h z^h$$

with coefficients in $\underset{\sim}{C}$ may now possibly converge for

$$0<|z|<\gamma$$

if γ is a sufficiently small positive number. If this is the case, then $f(z)$ becomes a single-valued analytic function for $|z|<\gamma$ with at most a pole at $z=0$.

It will be sufficient to study this convergence problem for basic S_p-series

$$h[z]=z^{-1}+\sum_{h=1}^{\infty} b_h z^h.$$

For by the results of the first chapter every S_p-series can be expressed in a trivial way in terms of a suitable basic series.

31. Let then $h[z]$ be any basic S_p-series with complex coefficients, and let further

$$(39)\qquad H(X,Y)=-(X^p-Y)(Y^p-X)+\sum_{r=0}^{p}\sum_{s=0}^{p} E_{rs}X^rY^s \qquad (E_{rs}=E_{sr})$$

be its polynomial. We shall apply the following result which is a special example from the theory of algebraic curves.

(60) *There exists a unique formal Laurent series*

$$U(X) = X^{1/p} + \sum_{h=1}^{\infty} u_h X^{-h/p}$$

with complex coefficients u_h *such that*

$$F(X, U(X)) \equiv 0.$$

PROOF. The equation for $U(X)$ has the explicit form

$$(X^p - U(X))(U(X)^p - X) = \sum_{r=0}^{p} \sum_{s=0}^{p} E_{rs} X^r U(X)^s, \tag{61}$$

where again

$$E_{pp} = E_{p-1,p} = E_{p,p-1} = 0$$

since $h[z]$ is basic.

Similarly as in previous proofs, denote by U_h any polynomial in $u_1, u_2, \cdots, u_h$ with complex coefficients; this polynomial may vary in different formulae.

Then evidently

$$X^p - U(X) = X^p - X^{1/p} - \sum_{h=1}^{\infty} u_h X^{-h/p},$$

$$U(X)^p - X = pu_1 X^{(p-2)/2} + \sum_{h=2}^{\infty} (pu_h + U_{h-1}) X^{(p-h-1)/p},$$

hence

$$(X^p - U(X))(U(X)^p - X) = pu_1 X^{(p^2+p-2)/p} + \sum_{h=2}^{\infty} (pu_h + U_{h-1}) X^{(p^2+p-h-1)/p}.$$

On the other hand,

$$X^r U(X)^s = X^{(pr+s)/p} + su_1 X^{(pr+s-2)/p} + \sum_{h=2}^{\infty} (su_h + U_{h-1}) X^{(pr+s-h-1)/p}.$$

Here the sum $pr + s$ for the terms $E_{rs} X^r U(X)^s$ with $E_{rs} \neq 0$ on the right-hand side of (61) does not exceed the value $p^2 + p - 2$ obtained for the pair of suffices $r = p$, $s = p - 2$. Hence, on comparing the coefficients of the different powers of $X^{1/p}$ on the two sides of (61), we obtain the following infinite system of recursive formulae,

$$pu_1 = E_{p,p-2}, \qquad pu_h = U_{h-1} \qquad \text{for} \qquad h \geqq 2,$$

which step by step determine all the Laurent coefficients u_h uniquely.

To the result (60) we can add the following corollary.

(62) *There exists a positive constant* Γ *such that*

$$|u_h| \leqq \Gamma^h \qquad (h = 1, 2, 3, \cdots).$$

For, if X runs over C, $U(X)$ defines an algebraic function of X in the neighbourhood of a branch point at infinity, and so the Laurent series for $U(X)$ necessarily converges for all sufficiently large finite values of $|X|$.

32. By definition, the algebraic equation

$$H(X, h[z]) = 0$$

has the $p+1$ roots

$$h[z^p] = z^{-p} + \cdots,$$

$$\text{and} \quad h[\varepsilon^j z^{1/p}] = \varepsilon^{-1} z^{-1/p} + \cdots \quad \text{for} \quad j = 0, 1, \cdots, p-1.$$

On replacing here X by Y and z by z^p, it follows similarly by the symmetry of $H(X, Y)$ that the equation

$$H(h[z^p], Y) = 0$$

has the $p+1$ roots

(63)
$$h[z^{p^2}] = z^{-p^2} + \cdots,$$
$$\text{and} \quad h[\varepsilon^j z] = \varepsilon^{-j} z^{-1} + \cdots \quad \text{for} \quad j = 0, 1, \cdots, p-1.$$

On the other hand,

$$H(X, U(X)) \equiv 0.$$

Here choose

$$X = h[z^p] = z^{-p} + \cdots,$$

so that

$$U(X) = X^{1/p} + \sum_{h=1}^{\infty} u_h X^{-h/p} = z^{-1} + \cdots$$

becomes a Laurent series in ascending powers of z beginning with the term z^{-1}. But $U(X)$ must be one of the series (63), and by the first terms of these series $U(X)$ is necessarily identical with $h[z]$.

It has thus been established that

(64)
$$h[z] = h[z^p]^{1/p} + \sum_{h=1}^{\infty} u_h h[z^p]^{-h/p}.$$

33. In this identity write

$$h[z^p] = z^{-p}(1+s) \quad \text{where} \quad s = \sum_{h=1}^{\infty} b_h z^{(h+1)p} = b_1 z^{2p} + b_2 z^{3p} + b_3 z^{4p} + \cdots.$$

Then for every real number r,

$$h[z^p]^r = z^{-pr} \sum_{n=0}^{\infty} \binom{r}{n} s^n.$$

It follows therefore from (64) that

$$h[z] = z^{-1} \sum_{n=0}^{\infty} \binom{1/p}{n} s^n + \sum_{h=1}^{\infty} \sum_{n=0}^{\infty} u_h z^h \binom{-h/p}{n} s^n. \tag{65}$$

The left-hand side of this identity is the Laurent series

$$h[z] = z^{-1} + \sum_{h=1}^{\infty} b_h z^h.$$

Also the right-hand side can be written as such a Laurent series. On comparing then again the coefficients of the different powers of z, we obtain a new representation of the Laurent coefficients b_h of $h[z]$.

It is, however, advisable first to break off both sides of (65) at a suitable finite power of z, and for this purpose we introduce the following notation.

Let M and N be any two positive integers. Then firstly, denote by

$$(s^n; M)$$

the polynomial in z which consists of all those terms in the formal power series for s^n that involve only the powers $1, z, z^2, \cdots, z^M$ of z. Secondly, let L_N be the sum

$$L_N = z^{-1} + \sum_{h=1}^{N} b_h z^h \tag{66}$$

of all terms on the left-hand side of (65) that involve only the powers

$$z^{-1}, z, z^2, \cdots, z^N$$

of z, and let R_N be the analogous sum of terms on the right-hand side of (65). It is obvious that

$$L_N = R_N, \tag{67}$$

and it further follows from (65) that

$$R_N = z^{-1} + z^{-1} \sum_{n=1}^{\infty} \binom{1/p}{n} (s^n; N+1) + \sum_{h=1}^{\infty} \sum_{n=0}^{\infty} u_h z^h \binom{-h/p}{n} (s^n; N-h). \tag{68}$$

Here the infinite sums can be replaced by finite ones. For the lowest term in the power series for s^n evidently is $b_1^n z^{2np}$, and hence

$$(s^n\,;M)=0 \quad \text{unless} \quad M \geqq 2np.$$

It follows therefore from (68) that also

$$R_N = z^{-1}+z^{-1}\sum_n \binom{1/p}{n}(s^n\,;N+1)+\sum_{h,n} u_h z^h \binom{-h/p}{n}(s^n\,;N-h) \tag{69}$$

where the sum Σ_n extends over all integers n satisfying

$$0 \leqq 2np \leqq N+1$$

and the double sum $\Sigma_{h,n}$ over all pairs of integers h, n such that

$$h \geqq 1, \quad n \geqq 0, \quad h+2np \leqq N.$$

34. The numbers b_h and u_h and the numerical coefficients that occur on the two sides of the equation (67) are real or complex numbers. Denote by the symbol * the operation which replaces all b_h and u_h and all these coefficients by their absolute values and which simultaneously substitute 1 for z. If * changes L_N and R_N into L_N^* and R_N^*, respectively, it is obvious that

$$L_N^* = 1+\sum_{h=1}^{N} |\, b_h \,| \tag{70}$$

and

$$L_N^* \leqq R_N^*. \tag{71}$$

An upper estimate for R_N^* can be obtained as follows. Denote by

$$[s^n\,;M]$$

the result of the operation * applied to $(s^n\,;M)$. By the definition of s,

$$s^n = \sum_{h_1=1}^{\infty}\cdots\sum_{h_n=1}^{\infty} b_{h_1}\cdots b_{h_n} z^{p(h_1+\cdots+h_n+n)},$$

whence

$$[s^n\,;M] = \sum_{h_1,\cdots,h_n} |\, b_{h_1}\cdots b_{h_n} \,| \tag{72}$$

where the multiple summation is extended over all sets of n positive integers $h_1,\cdots,h_n$ satisfying

$$p(h_1+\cdots+h_n+n) \leqq M.$$

The number of terms on the right-hand side of (72) is therefore equal to

$$\binom{[M/p]}{n} \leqq 2^{M/p},$$

and *the suffices* $h_1, \cdots, h_n$ *are all less than* M/p.

We know already that

(62) $$|u_h| \leqq \Gamma^h \qquad (h = 1, 2, 3, \cdots).$$

Assume that there further exists a constant C satisfying

(73) $$C \geqq 5 \quad \text{and} \quad C \geqq 2^{p-1}\Gamma^p + 1$$

such that

(74) $$|b_h| \leqq C^h \qquad \text{for} \quad 1 \leqq h \leqq N-1,$$

where N is a certain integer *not less than* 6. For $N = 6$ this assumption is certainly admissible.

35. Let

$$M \leqq pN,$$

an inequality which holds in particular when $M = N + 1$. The suffices $h_1, \cdots, h_n$ in (72) are smaller than M/p and hence cannot exceed $N - 1$. The hypothesis (74) may therefore be applied to all the coefficients b_h in the multiple sum (72).

Since this sum has at most $2^{M/p}$ terms, and since $h_1 + \cdots + h_n$ is for all these terms at most $(M/p) - n$, it follows that

$$[s^n ; M] \leqq 2^{M/p} \cdot C^{(M/p)-n} \quad \text{for} \quad 1 \leqq M \leqq pN.$$

By the definition of R_N^* the equation (69) implies then the upper estimate

$$R_N^* \leqq 1 + \sum_n \left| \binom{1/p}{n} \right| 2^{(N+1)/p} \cdot C^{((N+1)/p)-n} +$$
$$+ \sum_{h,n} \Gamma^h \left| \binom{-h/p}{n} \right| 2^{(N-h)/p} \cdot C^{((N-h)/p)-n}.$$

Here the sums Σ_n and $\Sigma_{h,n}$ are defined as in §33.

In the first sum,

$$\left| \binom{1/p}{n} \right| \leqq 1 \quad \text{for all } n, \quad \text{and} \quad \sum_{n=0}^{\infty} C^{-n} = \frac{C}{C-1} < 2$$

by the first assumption (73). Hence

(75) $$\sum_n \left| \binom{1/p}{n} \right| 2^{(N+1)/p} \cdot C^{((N+1)/p)-n} \leqq (2C)^{(N+1)/p} \sum_{n=0}^{\infty} C^{-n} < 2(2C)^{(N+1)/p}.$$

In the second sum,

$$\binom{-h/p}{n} \text{ has the same sign as } (-1)^n,$$

so that

$$\sum_{n=0}^{\infty} \left| \binom{-h/p}{n} \right| C^{-n} = \sum_{n=0}^{\infty} \binom{-h/p}{n} (-C)^{-n} = (1-C^{-1})^{-h/p}.$$

Therefore, by the second assumption (73),

$$\sum_{h,n} \Gamma \left| \binom{-h/p}{n} \right| 2^{(N-h)/p} \cdot C^{((N-h)/p)-n} \leqq \sum_{h=1}^{\infty} \Gamma^h (1-C^{-1})^{-h/p} \cdot (2C)^{(N-h)/p} =$$

(76)
$$= (2C)^{N/p} \sum_{h=1}^{\infty} (\Gamma^{-p} \cdot 2(C-1))^{-h/p} < (2C)^{N/p} \sum_{h=1}^{\infty} 2^{-h} = (2C)^{N/p}.$$

On combining the two upper estimates (75) and (76) with the earlier formulae (70) and (71), it follows finally that

$$L_N^* = 1 + \sum_{h=1}^{N} |b_h| \leqq R_N^* \leqq 1 + 2(2C)^{(N+1)/p} + (2C)^{N/p} < 1 + 3(2C)^{(N+1)/p},$$

whence, in particular,

(77)
$$|b_N| < 3(2C)^{(N+1)/p}.$$

36. This estimate can finally be replaced by

(78)
$$|b_N| < C^N,$$

provided it can be established that

$$3(2C)^{(N+1)/p} \leqq C^N.$$

This inequality will be satisfied if

$$3 \times 2^{(N+1)/p} \leqq C^{N-(N+1)/p},$$

or equivalently, if

$$C \geqq 3^{p[(p-1)N-1]^{-1}} \times 2^{(N+1)[(p-1)N-1]^{-1}}.$$

This formula is, in fact, true. For $N \geqq 6$ and $p \geqq 2$, hence

$$p[(p-1)N-1]^{-1} \leqq 1/2 \quad \text{and} \quad (N+1)[(p-1)N-1]^{-1} \leqq 3/2,$$

and further

$$3^{1/2} \times 2^{3/2} = 24^{1/2} < 5,$$

while, on the other hand, C is at least 5 by the first assumption (73).

The inequality (78) is therefore valid. Hence the induction hypothesis (74), which was assumed to be true for N, holds also for $N+1$. As it could be assumed for $N=6$, it holds then for every positive integer N, and so

(79) $$|b_h| \leqq C^h \qquad (h = 1, 2, 3, \cdots).$$

37. These inequalities (79) mean that *the formal Laurent series* $h[z]$ *is in fact convergent for*

$$0 < |z| < 1/C.$$

In other words, $h[z]$ *is a single-valued analytic function for* $|z| < 1/C$ *with a simple pole of residue* 1 *at* $z=0$.

This result has been proved for every formal *basic* S_p*-series* with complex coefficients. It can immediately be generalised as follows.

THEOREM 8. *Let* $f(z)$ *be an arbitrary formal* S_p*-series over* C. *Then there exists a positive constant* γ *such that* $f(z)$ *converges for* $0<|z|<\gamma$ *and hence represents a single-valued analytic function for* $|z|<\gamma$. *The origin* $z=0$ *is a pole of* $f(z)$ *if* $f(z)$ *is of negative order, but is a regular point if the order of* $f(z)$ *is non-negative.*

PROOF. If the order m of $f(z)$ is negative, then by §15 there exist a basic S_p-series $h[z]$ and two constants $C_0 \neq 0$ and C_1 such that $f(z) = C_0 h[z^{-m}] + C_1$. If the order m of $f(z)$ is positive, then Theorem 3 shows that $f(z)^{-1}$ is an S_p-series of the negative order $-m$. Hence in this case $f(z)$ itself can be written in the form

$$f(z) = (C_0 h[z^m] + C_1)^{-1}.$$

If, finally, $f(z)$ has the order 0 and a_0 is its constant term, then $f(z) - a_0$ is an S_p-series of a certain positive order m, and so $f(z)$ allows a representation

$$f(z) = (C_0 h[z^m] + C_1)^{-1} + a_0.$$

Our general convergence result for basic S_p-series implies then the assertion since in the last two cases $f(z)$ can be expressed as a power series convergent in a neighbourhood of $z=0$.

38. If $f(z)$ is any S_p-series over C, then the same symbol $f(z)$ is used to denote the analytic function which is defined by the series in its region of convergence is from here extended into its whole domain of existence by analytic continuation. Such an analytic function is called an S_p-function. In the special case when $f(z) = h[z]$ is a basic S_p-series, we speak of basic S_p-functions. It would evidently suffice to study basic S_p-functions.

There are basic S_p-functions which exist and are single-valued in the whole z-plane; for instance, the rational functions z^{-1} and $z^{-1}+z$ are for every prime p of this kind. I do not know whether there are also non-rational algebraic basic S_p-functions. On the other hand, I conjecture that there exist transcendental basic S_p-functions which can be continued into the whole z-plane, but may be multi-valued.

There do exist basic S_p-functions which can be defined only in the unit circle

$$U : |z| < 1,$$

and which in U are regular and single-valued except for the pole at $z = 0$. A particularly important example of this kind of function can be derived from the modular function

$$j(\omega) = \sum_{h=-1}^{\infty} a_h e^{2pi\omega} \qquad (a_{-1} = 1, a_0 = 744, \text{etc.})$$

by putting

$$z = e^{2\pi i\omega},$$

and subtracting from $j(\omega)$ its constant term $a_0 = 744$. The transformation equations for $j(\omega)$ imply that the derived Laurent series

$$g[z] = z^{-1} + \sum_{h=1}^{\infty} a_h z^h$$

defines for every prime p a basic S_p-function. Since $j(\omega)$ exists only in the complex upper halfplane, $g[z]$ is defined in U, but has the circle $|z| = 1$ as its natural boundary. It is single-valued and regular in U except for its pole at $z = 0$.

39. For arbitrary S_p-functions only a weaker result can be proved.

THEOREM 9. *Every basic S_p-function can be continued into the whole of U, but may become multi-valued by having a certain set of branch points in U, all of finite orders. The function is regular at every point distinct from its pole $z = 0$ and from these branch points.*
The same result holds for general S_p-functions of negative orders. For general S_p-functions of non-negative orders the same result remains valid except that now there is no pole at $z = 0$, but there may be poles elsewhere.

PROOF. It evidently suffices to prove the assertion for basic S_p-functions $h[z]$. In this case we know already that the only singularity of $h[z]$ in a certain region $|z| < 1/C$ is the pole at $z = 0$.

Let the assertion be false. There exists then a largest open disc where $0 < \rho < 1$, in which the assertion is still true. We shall prove that this assumption leads to a contradiction by constructing a larger open disc in which $h[z]$ has still the same properties.

For this purpose denote by ρ' a constant satisfying

$$\rho'^{p} < \rho < \rho' < 1,$$

and let then V_1 and V_2 be the discs

$$V_1 : |z| \leqq \rho'^{p} \quad \text{and} \quad V_2 : |z| \leqq \rho',$$

respectively. Thus V_1 is contained in V, and V is contained in V_2.

If $H(X, Y)$ is again the polynomial of $h[z]$, denote by $A(X)$ the algebraic function of the complex variable X which is defined by

$$H(X, A(X)) = 0.$$

(The formal Laurent series $U(X)$ of §31 defines thus one of the branches of $A(X)$ at the point of infinity.) Since $H(X, Y)$ is monic in X, $A(X)$ is regular for finite X except at its branch points, but also these branch points are not poles of $A(X)$.

We know that

$$H(h[z], h[z^{p}]) = 0$$

identically in z. The definition of $A(X)$ implies therefore that

$$h[z] = A(h[z^{p}]).$$

Let now z run over the closed disc V_2 so that z^{p} runs over the closed disc V_1 and hence lies in a closed subset of V. By the hypothesis about V the only singularities of $h[z^{p}]$ are then its pole at $z = 0$ and a certain set of branch points of finite orders; here this set can consist at most of finitely many branch points. Since $h[z]$ is an algebraic function of $h[z^{p}]$, it has the same properties for z in V_2. But by the definition of ρ', V is a proper subset of the interior $|z| < \rho'$ of V_2. This implies our assertion and proves the theorem.

40. For the special case when $p = 2$ and $H(X, Y)$ is reducible, we shall now construct a basic S_p-function $h[z]$ which is multi-valued and in fact has infinitely many branch points in U.

By the remark to the formula (37) of §20 the polynomial $H(X, Y)$ must have the form

$$H(X, Y) = -(X^2 - Y + c)(Y^2 - X + c);$$

here c lies in C. For simplicity, let c be a *positive* constant.

In the present case the basic function

$$h[z] = z^{-1} + \sum_{h=1}^{\infty} b_h z^h$$

satisfies the functional equation

$$h[z]^2 - h[z^2] + c = 0$$

from which it easily follows that $h[z]$ is an odd function, hence that

$$b_{2h} = 0 \quad \text{for} \quad h = 1, 2, 3, \cdots.$$

We further find by the usual method that

$$b_1 = -c/2 \quad \text{and} \quad 2b_{2h+1} = -\sum_{j=0}^{h-1} b_{2j+1} b_{2h-2j-1} + b_h \quad \text{for} \quad h \geqq 1.$$

These formulae show in particular that all the coefficients b_{2h+1} are negative. Thus, on taking only the term with $j = 0$ in the sum for $2b_{2h+1}$, it follows that

$$2b_{2h+1} \leqq -b_1 b_{2h-1} \quad \text{if} \quad h \geqq 1$$

and therefore that

$$b_{2h+1} \leqq -(c/4)^h \qquad (h = 0, 1, 2, \cdots).$$

Hence, if z is positive and the left-hand series converges,

$$\sum_{h=1}^{\infty} b_h z^h = \sum_{h=0}^{\infty} b_{2h+1} z^{2h+1} \leqq -2z \sum_{h=0}^{\infty} (cz^2/4)^h = -8z(4 - cz^2)^{-1}.$$

Denote by γ the radius of convergence of $\Sigma_{h=1}^{\infty} b_h z^h$. This estimate for the series, and the fact that γ is positive by Theorem 8, imply that

$$0 < \gamma < 2c^{-1/2}.$$

Choose therefore $c > 4$; then

$$0 < \gamma < 1.$$

Since all the coefficients b_h are negative or zero, the point $z = \gamma$ on the positive real axis is a singular point of $h[z]$, but by $\gamma^2 < \gamma$ naturally is a regular point of $h[z^2]$. By Theorem 9, $z = \gamma$ necessarily is a branch point of $h[z]$. The functional equation requires then that

$$h[\gamma^2] = c.$$

Therefore $h[z^2] - c$ has a convergent development of the form

$$h[z^2] - c = \sum_{h=1}^{\infty} c'_h (z - \gamma)^h,$$

where $c_1' \neq 0$, in a certain neighbourhood of $z = \gamma$. Hence, by the functional equation, $h[z]$ allows in a possibly smaller neighbourhood of $z = \gamma$ a convergent development

$$h[z] = \sum_{h=1}^{\infty} c_h (z - \gamma)^{h/2}$$

where $c_1 \neq 0$.

On applying now the functional equation repeatedly, we see that also the infinitely many points

$$e^{2\pi i m \cdot 2^{-n}} \gamma^{2-n} \qquad \begin{pmatrix} m = 0, 1, \cdots, 2^n - 1 \\ n = 0, 1, 2, \cdots \end{pmatrix}$$

are branch points of $h[z]$, possibly of higher order. These branch points converge to every point of $|z| = 1$. This circle forms therefore the natural boundary of $h[z]$.

The same proof leads to the following result.

THEOREM 10. *A basic S_p-series $h[z]$ defines a basic S_p-function with at least one branch point inside the unit circle if and only if its radius of convergence is less than* 1.

For there is again at least one singular point of $h[z]$ on the circle of convergence, and if its radius is less than 1, then by Theorem 9 this singular point must be a branch point.

Just as in the example, if $h[z]$ has at least one branch point inside $|z| = 1$, it probably has an infinite sequence of branch points tending to this circle which therefore becomes again a natural boundary.

Unfortunately, the actual determination of the radius of convergence of a general basic S_p-series $h[z]$ when its polynomial $H(X, Y)$ is given does not seem to be an easy problem.

Chapter 4. **Modular functions and S_p-functions**

41. In this final chapter we shall study relations between the theory of S_p-functions and that of certain modular functions.

To begin with, let

$$j(\omega) = \sum_{h=-1}^{\infty} a_h z^{2\pi i h \omega} \qquad (a_{-1} = 1)$$

be the modular function of level 1. Put

$$z = e^{2\pi i \omega}$$

and

$$f(z) = z^{-1} + \sum_{h=0}^{\infty} a_h z^h \quad \text{and} \quad h[z] = z^{-1} + \sum_{h=1}^{\infty} a_h z^h,$$

$$\text{so that } f(z) = h[z] + a_0.$$

Hence $f(z)$ is a normed S_p-function of order -1, and $h[z]$ is a basic S_p-function, both for every prime p. Both functions are regular for $|z| \nless 1$, but have the circle $|z| = 1$ as their natural boundaries; the origin is a simple pole of $f(z)$ and $h[z]$.

One has computed the coefficients a_h for all suffices $h \leqq 500$. Here are a few of the first coefficients.

$a_0 = 744$	$a_7 = 4465,69940,71935$
$a_1 = 1,96884$	$a_8 = 40149,08866,56000$
$a_2 = 214,93760$	$a_9 = 3,17644,02297,84420$
$a_3 = 8642,99970$	$a_{10} = 22,56739,33095,93600$
$a_4 = 2,02458,56256$	$a_{11} = 146,21191,14995,19294$
$a_5 = 33,32026,40600$	$a_{12} = 874,31371,96857,75360$
$a_6 = 425,20233,00096$	

For every prime p, the function $j(\omega)$ satisfies the transformation equation

$$F_p(j(p\omega), j(\omega)) = 0,$$

where $F_p(X, Y)$ is a symmetric polynomial of degree $p+1$ in X and Y which is monic in both. Hence $f(z)$ satisfies the functional equation

$$F_p(f(z^p), f(z)) = 0$$

and is by Theorem 1 a normed S_p-function with the polynomial $F_p(X, Y)$. Similarly, $h[z]$ is a basic S_p-function.

Since we may substitute the a_h for the b_h, the coefficients a_h satisfy for every prime p the corresponding infinite system of recursive formulae which allows successively to determine all the coefficients with $h \geqq p^2 + p$ as polynomials in these coefficients with $1 \leqq h \leqq p^2 + p - 1$; in addition, the latter satisfy $[p(p-1)]/2$ conditions.

For the lowest cases, we found for $p = 2$ the recursive formulae (46) together with the one condition (44), and for $p = 3$ the recursive formulae (51) together with three conditions (48). For the next two cases $p = 5$ and $p = 7$ we determined only the recursive formulae for the coefficients with suffices $5h$ and $7h$, respectively, as well as a few other coefficients.

Included in these recursive formulae and conditions are in particular the following simple equations,

$$a_4 = a_3 + (a_1^2 - a_1)/2, \qquad a_9 = a_5 + a_1a_3 + a_2^2 + (a_1^3 - a_1)/3,$$

$$a_6 = a_4 + a_1a_2, \qquad a_{10} = a_8 + a_1a_4 + a_2a_3,$$

$$a_8 = a_5 + a_1a_3 + (a_2^2 - a_2)/2, \quad a_{12} = a_6 + a_1a_4 + 2a_2a_3 + a_1^2a_2.$$

From the earlier table of the numerical values of the coefficients they can easily be checked.

It needs scarcely be stated that the recursive formulae for $p = 2$ are useful for the computation of the coefficients a_h of $j(\omega)$, i.e., of $h[z]$.

42. Denote by $K(p)$ the set of all recursive formulae and conditions that hold for the prime p and for every basic S_p-function $h[z]$. The function $h[z]$ derived from $j(\omega)$ is an S_p-function for every prime p and is basic, hence satisfies $K(p)$ for every prime p. *Therefore the infinitely many sets of formulae*

$$K(2), K(3), K(5), K(7), \cdots$$

all hold for this special function $h[z]$ and so are consistent with one another.

Our original investigation of basic S_p-series and S_p-functions involved only one set $K(p)$ belonging to a *single* prime p. It is then rather surprising that the existence of the modular function $j(\omega)$ and its property of satisfying all the transformation equations of prime orders should imply certain relations between the different sets $K(p)$. It would have great interest to study these relations.

43. The theory of S_p-functions allows to prove two theorems that explain in how far the modular function $j(\omega)$ is determined by its transformation equations of any prime order.

THEOREM 11. *Let p be a prime and $F_p(X, Y)$ the transformation polynomial of $j(\omega)$ of order p. Let further*

$$\phi(z) = z^{-1} + \sum_{h=0}^{\infty} \alpha_h z^h$$

be a formal Laurent series with coefficients in C such that

$$F_p(\phi(z^p), \phi(z)) = 0.$$

Then $\phi(z)$ converges and defines a single-valued regular function for $0 < |z| < 1$, and

$$j(\omega) = \phi(e^{2\pi i\omega})$$

identically for all ω in the complex upper halfplane.

PROOF. The polynomial $F_p(X, Y)$ is symmetric, monic, and of the exact degree $p+1$ in X and Y. Further the formal series

$$\phi(z^p), \quad \text{and} \quad \phi(\varepsilon^j z^{1/p}) \quad \text{where} \quad j = 0, 1, \cdots, p-1,$$

obviously are all distinct. It follows then from Theorem 1 that $\phi(z)$ is a normed S_p-series of order -1, and that $F_p(X, Y)$ is its polynomial. But the series $f(z)$ defined in §41 had exactly the same properties. Hence, by §12 and by Theorem 4, the two series $\phi(z)$ and $f(z)$ are identical, whence the assertion.

THEOREM 12. *Let p be a prime, and let*

$$\phi(z) = z^{-1} + \sum_{h=0}^{\infty} \alpha_h z^h$$

be a formal S_p-series such that

$$\alpha_h = a_h \quad \textit{for} \quad 0 \leqq h \leqq p^2 + p - 1,$$

where the a_h are the Fourier coefficients of $j(\omega)$ as defined in §41. *Then $\phi(z)$ converges and defines a single-valued function for $0 < |z| < 1$, and*

$$j(\omega) = \phi(e^{2\pi i \omega})$$

identically for all ω in the complex upper halfplane.

PROOF. The coefficients α_h of $f(z)$ coincide for $0 \leqq h \leqq p^2 + p - 1$, from which it follows that the polynomials $\Phi(X, Y)$ of $\phi(z)$ and $F_p(X, Y)$ of $f(z)$ are identical. For both polynomials are determined uniquely by the same set of coefficients $\alpha_h = a_h$ with $0 \leqq h \leqq p^2 + p - 1$. The assertion is therefore contained in Theorem 11.

A completely different characterisation of $j(\omega)$ by its transformation equations was given by Siegel, (1964). It depended on the behaviour of $j(\omega)$ near fixed points of elliptic transformations.

44. There are modular functions of higher level which for suitable primes p lead to S_p-functions.

As a first example consider the cube root of $j(\omega)$. In the notation of H. Weber, (1908), p. 179, put

$$\gamma_2(\omega) = j(\omega)^{1/3} = h[z] = z^{-1} + \sum_{k=0}^{\infty} b_{3k+2} z^{3k+2},$$

where now, in the series for $h[z]$, z denotes the cube root of the previous z,

$$z = e^{2\pi i \omega/3},$$

and where the first coefficients b_{3k+2} have the values

$$b_2 = 248, \quad b_5 = 4124, \quad b_8 = 34752.$$

By Weber, (1908, l.c. p. 248), $h[z]$ certainly is a basic S_2-function. For, as he shows, the three functions $h[z^2]$, $h[z^{1/2}]$, and $h[-z^{1/2}]$ are roots of

$$F(X, h[z]) = 0,$$

where $F(X, Y)$ in the present case denotes the symmetrical polynomial

$$F(X, Y) = -(X^2 - Y)(Y^2 - X) + 496XY - 54000 = 0.$$

Since $h[z]$ is an S_2-function, its coefficients b_h must satisfy the former condition (44) and the recursive formulae (46) of §§24–25. Here (44) trivially is valid since b_h always vanishes when $h \not\equiv 2 \pmod 3$. For the same reason, the recursive formulae (46) take now the following simpler form:

(79a) $$b_{12k+2} = b_{6k+2} + \sum_{j=0}^{k-1} b_{3j+2} b_{6k-3j-1},$$

(79b) $$b_{12k+5} = b_{6k+5} + \sum_{j=0}^{k-1} b_{3j+2} b_{6k-3j+2} - \sum_{j=0}^{2k-1} (-1)^{j-1} b_{3j+2} b_{12k-3j+2} + \\ + \sum_{j=0}^{k-1} b_{3j+2} b_{12k-12j-4} - b_2 b_{6k+2} + (b_{3k+2}^2 - b_{3k+2})/2 + \\ + (b_{6k+2}^2 + b_{6k+2})/2,$$

(79c) $$b_{12k+8} = b_{6k+5} + \sum_{j=0}^{k-1} b_{3j+2} b_{6k-3j+2} + (b_{3k+2}^2 - b_{3k+2})/2,$$

(79d) $$b_{12k+11} = b_{6k+8} + \sum_{j=0}^{k} b_{3j+2} b_{6k-3j+5} - \sum_{j=0}^{2k} (-1)^{j-1} b_{3j+2} b_{12k-3j+8} + \\ + \sum_{j=0}^{k} b_{3j+2} b_{12k-12j+2} - b_2 b_{6k+5} - (b_{6k+5}^2 - b_{6k+5})/2.$$

Here as usual empty sums mean 0, and k may be any non-negative integer.

These recursive formulae can again be used to evaluate any number of coefficients b_{3k+2} of $h[z]$. For the lowest suffices,

$$b_{11} = 2{,}13126, \quad b_{14} = 10{,}57504, \quad b_{17} = 45{,}30744, \quad b_{20} = 173{,}33248.$$

It would be interesting to decide whether $h[z]$ is an S_p-function for all primes $p \equiv 2 \pmod 3$. The function in fact has this property when $p = 5$, as follows from the form of the transformation equation between $j(\omega)$ and $j(5\omega)$ as given in R. Fricke, (1922), p. 393.

45. As a second example consider the Jacobi module

$$u(\omega) = k^{1/4} = 2^{1/2} q^{1/8} \prod_{m=1}^{\infty} \frac{1+q^{2m}}{1+q^{2m-1}} =$$

$$= 2^{1/2} q^{1/8}(1 - q + 2q^2 - 3q^3 + 4q^4 - 6q^5 + 9q^6 - \cdots),$$

where

$$q = e^{\pi i \omega}.$$

Put

$$z = e^{\pi i \omega/8} \quad \text{and} \quad h[z] = (4/k)^{1/4},$$

so that

$$h[z] = z^{-1} \prod_{m=1}^{\infty} \frac{1+z^{16m-8}}{1+z^{16m}} = z^{-1} + \sum_{k=0}^{\infty} b_{8k+7} z^{8k+7},$$

where the lowest coefficients are

$$b_7 = b_{63} = +1, \quad b_{15} = b_{47} = b_{55} = b_{79} = -1, \quad b_{23} = b_{31} = b_{39} = b_{71} = 0.$$

Let now p be a prime satisfying

$$p \equiv \pm 1 \pmod 8.$$

Then $v(\omega) = u(p\omega)$ is connected with $u(\omega)$ by a symmetric transformation equation which in the lowest case $p = 7$ has the form (Fricke, l.c., p. 501)

$$u^8 + v^8 - uv(8u^6v^6 - 28u^5v^5 + 56u^4v^4 - 70u^3v^3 + 56u^2v^2 - 28uv + 8) = 0.$$

It follows from Theorem 1 that u is an R_7-function of order $+1$. By §9, this explains why in this equation all terms different from $u^7 + v^7$ are divisible by uv.

We deduce by a trivial change of variables in this equation that the polynomial $H(X, Y)$ of $h[z]$ is given by

$$F(X, Y) = -(X^7 - Y)(Y^7 - X) +$$

$$+ 7(X^6Y^6 - 4X^5Y^5 + 10X^4Y^4 - 16X^3Y^3 + 16X^2Y^2 - 9XY) = 0.$$

There are thus particularly simple explicit expressions as polynomials in $h[z]$ for the elementary symmetric functions $s_1, s_2, \cdots, s_8$ of the elements of Σ_h. These would allow again to derive recursive formulae for the coefficients b_{8k+7}.

It is clear that $h[z]$ is a basic R_p-function for all the primes $8n \pm 1$.

46. As a final example consider the Schlaefli modular equations which concern the function (Fricke, l.c., pp. 502–8)

$$s(\omega) = 2^{1/3}(kk')^{1/12} = 2^{1/2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1})^{-1} =$$

$$= 2^{1/2} q^{1/24}(1 - q + q^2 - 2q^3 + 2q^4 - 3q^5 + 4q^6 - \cdots),$$

where again $q = e^{\pi i \omega}$. Put now

$$z = e^{\pi i \omega/24} \quad \text{and} \quad h[z] = 2^{1/2} s(\omega)^{-1},$$

so that

$$h[z] = q^{-1/24} \prod_{n=0}^{\infty} (1 + q^{2n+1}) = z^{-1} + \sum_{k=0}^{\infty} b_{24k+23} z^{24k+23},$$

with the lowest coefficients

$$b_{23} = b_{71} = b_{95} = b_{119} = b_{143} = b_{167} = +1, \qquad b_{47} = 0,$$

$$b_{191} = b_{215} = b_{239} = b_{263} = +2.$$

Denote by p any prime not less than 5. Then the functions $s(\omega)$ and $t(\omega) = s(p\omega)$ are connected by a symmetric algebraic equation with the highest terms s^{p+1} and t^{p+1}. Hence Theorem 1 leads again easily to the result that in the present case $h[z]$ is a basic S_p-function (and even a basic R_p-function) for all such primes.

For the lowest primes $p = 5$ and $p = 7$ the transformation equations are

$$s^6 + t^6 + st(s^4t^4 - 4) = 0 \quad \text{and} \quad s^8 + t^8 - st(s^6t^6 - 7s^3t^3 + 8) = 0,$$

respectively. This means that for $p = 5$ the polynomial $H(X, Y)$ of $h[z]$ is equal to

$$H(X, Y) = -(X^5 - Y)(Y^5 - X) + 5XY,$$

and for $p = 7$ equal to

$$H(X, Y) = -(X^7 - Y)(Y^7 - X) + 7(X^4Y^4 - XY).$$

Of particular interest is the simple result for $p = 5$. From the explicit form of $H(X, Y)$ the elementary symmetric functions of the elements of Σ_h are given by

$$s_1 = h[z]^5, \quad s_2 = s_3 = s_4 = 0, \quad s_5 = -h[z], \quad s_6 = h[z]^6.$$

By means of these formulae it would again not be difficult, but rather tedious, to derive recursive formulae for the coefficients b_{24k+23}. But since evidently b_{24k+23} is equal to the number of partitions of $k + 1$ into distinct odd integers, there is no need to do so.

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