On a Class of Transcendental Decimal Fractions

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Almost forty years ago, I proved (cf. [1], [2]) that the decimal fraction

$$0.123456789101012\cdots$$

is transcendental. In the present paper, this result will be generalized, as follows.

Denote by $\alpha(n)$ an arbitrary positive integral-valued arithmetic function. Write successively after the decimal point

each of the 1-digit numbers $1, 2, \cdots, 9$, each $\alpha(1)$ times repeated,
each of the 2-digit numbers $10, 11, \cdots, 99$, each $\alpha(2)$ times repeated,
each of the 3-digit numbers $100, 101, \cdots, 999$, each $\alpha(3)$ times repeated,
etc.

It will be proved that the resulting decimal fraction is transcendental.

Since this slight generalization makes perhaps the method of proof a little clearer, we establish the analogous theorem for fractions to an arbitrary integral basis $q \geqq 2$.

§1. Let $q \geqq 2$ be a fixed integer and put

$$x = 1/q\,.$$

If

$$a = \{a_1, a_2, a_3, \cdots\}$$

denotes a fixed sequence of positive integers, put

$$a(0) = 0, \qquad a(n) = a_1 + a_2 + \cdots + a_n \quad \text{for} \quad n \geqq 1\,,$$

and for $n \geqq 1$,

$$A_n = \sum_{h=1}^{n} h(a(q^h - 1) - a(q^{h-1} - 1))$$
$$= na(q^n - 1) - (a(q-1) + a(q^2 - 1) + \cdots + a(q^{n-1} - 1))\,.$$

By digits to the basis q we mean any one of the numbers $0, 1, 2, \cdots, q-1$. Instead of the ordinary decimal expansion we shall be concerned with expansions to the basis q, and we shall write symbolically

$$d_{-m}d_{-m+1}\cdots d_{-1}d_0\cdot d_1d_2d_3\cdots \quad \text{for} \quad \sum_{h=-m}^{\infty} d_h x^h\,;$$

here the d_h denote digits.

§**2.** Let in particular

$$\sigma(a)=0\cdot d_1d_2d_3\cdots$$

be the expansion to the basis q in which we have written successively after the point the expansions to the basis q of the integers

$$1,\ a_1 \text{ times repeated},$$

$$2,\ a_2 \text{ times repeated},$$

$$3,\ a_3 \text{ times repeated},$$

etc.; thus $d_1, d_2, d_3, \cdots$, are the resulting digits of $\sigma(a)$. We begin by writing this number as a rapidly convergent series.

In their expansions to the basis q, the integers from 1 to $q-1$ have exactly one digit, those from q to q^2-1 have exactly two digits, those from q^2 to q^3-1 have exactly three digits, etc. Hence in the expansion of $\sigma(a)$ there are after the point

$$a_1+a_2+\cdots+a_{q-1}=a(q-1)$$

single digits which correspond to 1-digit integers; next there are

$$2(a_q+a_{q+1}+\cdots+a_{q^2-1})=2(a(q^2-1)-a(q-1))$$

digits in pairs which correspond to 2-digit integers; following this, there are

$$3(a_{q^2}+a_{q^2+1}+\cdots+a_{q^3-1})=3(a(q^3-1)-a(q^2-1))$$

digits in sets of three which correspond to 3 digit integers, etc.

By its definition in Section 1, A_n is then the total number of digits in $\sigma(a)$ after the point derived from integers which, to the basis q, have at most n

digits. Here the first integer which, to the basis q, has n digits is

$$q^{n-1} = 100 \cdots 00,$$

where there are $n-1$ digits 0. In the expansion of $\sigma(a)$ it evidently occurs with the factor

$$x^{A_{n-1}+n}.$$

§3. Now put

$$s_n = \sum_{k=q^{n-1}}^{q^n-1} k(x^{n\{a(k-1)+1\}} + x^{n\{a(k-1)+2\}} + \cdots + x^{na(k)}), \qquad n = 1, 2, 3, \cdots,$$

and in particular,

$$s_1 = \sum_{k=1}^{q-1} k(x^{a(k-1)+1} + x^{a(k-1)+2} + \cdots + x^{a(k)}).$$

The terms in s_n in the lowest and the highest powers of x are

$$q^{n-1} \cdot x^{n\{a(q^{n-1}-1)+1\}} \quad \text{and} \quad (q^n - 1) \cdot x^{na(q^n-1)},$$

respectively. Hence, for $n \geqq 2$, the first and the last terms of the product

$$x^{(A_{n-1}+n)-n\{a(q^{n-1}-1)+1\}} s_n = t_n \quad \text{say},$$

are

$$q^{n-1} \cdot x^{A_{n-1}+n} \quad \text{and} \quad (q^n - 1) \cdot x^{A_n},$$

respectively.

It follows that s_1 is the sum of all the contributions to $\sigma(a)$ from the 1-digit integers and similarly, for $n \geqq 2$, t_n is the sum of all the contributions to $\sigma(a)$ from the n-digit integers. Consequently

$$\sigma(a) = s_1 + \sum_{n=2}^{\infty} t_n.$$

Here, for $n \geqq 2$, by the definition of A_n in Section 1,

$$\begin{aligned}(A_{n-1}+n) - n(a(q^{n-1}-1)+1) &= A_{n-1} - na(q^{n-1}-1) \\ &= -(a(q-1) + a(q^2-1) + \cdots + a(q^{n-1}-1)).\end{aligned}$$

Hence, if for $n \geqq 2$ we put

$$e(n) = a(q-1) + a(q^2-1) + \cdots + a(q^{n-1}-1),$$

we have shown that

$$\sigma(a) = s_1 + \sum_{n=2}^{\infty} x^{-e(n)} s_n .$$

§**4.** In this expansion, the sums s_n can be replaced by more explicit expressions. As a geometric series,

$$x^{n\{a(k-1)+1\}} + x^{n\{a(k-1)+2\}} + \cdots + x^{na(k)} = \frac{x^n}{1-x^n}(x^{na(k-1)} - x^{na(k)}),$$

so that

$$s_n = \frac{x^n}{1-x^n} \sum_{k=q^{n-1}}^{q^n-1} k(x^{na(k-1)} - x^{na(k)}),$$

or equivalently,

$$s_n = \frac{x^n}{1-x^n}\left(q^{n-1}x^{na(q^{n-1}-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{n-1}}^{q^n-1} x^{na(k)}\right).$$

It follows therefore that

$$\begin{aligned} \sigma(a) = {} & \frac{x}{1-x}\left(1 - qx^{a(q-1)} + \sum_{k=1}^{q-1} x^{a(k)}\right) \\ & + \sum_{n=2}^{\infty} \frac{x^{n-e(n)}}{1-x^n}\left(q^{n-1}x^{na(q^{n-1}-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{n-1}}^{q^n-1} x^{na(k)}\right). \end{aligned} \tag{1}$$

In a special case, to which we now proceed, this somewhat involved formula can be further simplified.

§**5.** For this purpose we assume from now on that, for every positive integer n, all the integers

$$a_k, \quad \text{where} \quad q^{n-1} \leqq k \leqq q^n - 1,$$

have one and the same value, the value $\alpha(n)$ say; here $\alpha(n)$ is a positive integer-valued function of n which is not otherwise restricted.

We find now easily that

$$a(k)=\alpha(1)k \qquad \text{for} \qquad 1 \leqq k \leqq q-1 ,$$

$$a(k)=(\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2})(q-1)+\alpha(n)(k-q^{n-1}+1)$$

$$\text{for} \qquad q^{n-1} \leqq k \leqq q^n-1,\ n \geqq 2 .$$

Thus, in particular, for $n=1,2,3,\cdots$,

$$a(q^n-1)=(\alpha(1)+\alpha(2)q+\cdots+\alpha(n)q^{n-1})(q-1) ,$$

whence

$$e(n)=((n-1)\alpha(1)+(n-2)\alpha(2)q+\cdots+1\cdot\alpha(n-1)q^{n-2})(q-1) \quad \text{for} \quad n \geqq 2 .$$

The explicit expressions for $a(k)$ imply next that

$$\sum_{k=q^{n-1}}^{q^n-1} x^{na(k)} = \sum_{k=q^{n-1}}^{q^n-1} x^{n(\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)+\alpha(n)(k-q^{n-1}+1))}$$

$$= x^{n\cdot\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)} \cdot x^{n\alpha(n)} \cdot \frac{1-x^{n\alpha(n)(q-1)q^{n-1}}}{1-x^{n\alpha(n)}} ,$$

while moreover,

$$q^{n-1}x^{na(q^{n-1}-1)} = q^{n-1}x^{n\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)} ,$$

$$q^n x^{na(q^n-1)} = q^n x^{n\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)} \cdot x^{n\alpha(n)(q-1)q^{n-1}} .$$

Therefore,

$$q^{n-1}x^{na(q^{n-1}-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{n-1}}^{q^n-1} x^{na(k)} = x^{n\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)}$$

$$\times\left(\frac{q^{n-1}-q^{n-1}x^{n\alpha(n)}+x^{n\alpha(n)}}{1-x^{n\alpha(n)}} - \frac{q^n-q^n x^{n\alpha(n)}+x^{n\alpha(n)}}{1-x^{n\alpha(n)}}\, x^{n\alpha(n)(q-1)q^{n-1}}\right) .$$

It follows that

$$s_n = \frac{x^n}{1-x^n}\,\frac{x^{n\alpha(n)}}{1-x^{n\alpha(n)}}\, x^{n\{\alpha(1)+\alpha(2)q+\cdots+\alpha(n-1)q^{n-2}\}(q-1)}$$

$$\times((q^{n\alpha(n)+n-1}-q^{n-1}+1)-(q^{n\alpha(n)+n}-q^n+1)x^{n\alpha(n)(q-1)q^{n-1}}) ;$$

here we have applied the equation $x=1/q$.

On substituting this value of s_n in (1), we find that

$$\sigma(a)=\frac{x^{\alpha(1)+1}}{(1-x)(1-x^{\alpha(1)})}(q^{\alpha(1)}-(q^{\alpha(1)+1}-q+1)x^{\alpha(1)(q-1)})$$

$$+\sum_{n=2}^{\infty}\frac{x^{n\alpha(n)+n}}{(1-x^n)(1-x^{n\alpha(n)})}$$

$$\times((q^{n\alpha(n)+n-1}-q^{n-1}+1)-(q^{n\alpha(n)+n}-q^n+1)x^{n\alpha(n)(q-1)q^{n-1}})$$

$$\times x^{\{1\cdot\alpha(1)+2\cdot\alpha(2)q+\cdots+(n-1)\alpha(n-1)q^{n-2}\}(q-1)}.$$

This formula can finally be simplified by taking together the positive part of the n-th term with the negative part of the $(n-1)$-st term throughout. Thus we arrive at the following simple expansion where we have once more used the fact that $x=1/q$:

$$\sigma(a)=\frac{q^{\alpha(1)}}{(q-1)(q^{\alpha(1)}-1)}$$

(2)
$$-\sum_{n=1}^{\infty}\left(\frac{q^{n\alpha(n)+n}-q^n+1}{(q^n-1)(q^{n\alpha(n)}-1)}-\frac{q^{(n+1)\alpha(n+1)+n}-q^n+1}{(q^{n+1}-1)(q^{(n+1)\alpha(n+1)}-1)}\right)$$

$$\times q^{-\{\alpha(1)+2\alpha(2)q+\cdots+n\alpha(n)q^{n-1}\}(q-1)}.$$

In the special case when

$$\alpha(n)=1 \quad \text{for} \quad \text{all} \quad n,$$

this development of $\sigma(a)$ reduces to a formula which I obtained almost forty years ago in [1], [2]. (See also Nicholson [5] and my recent note [4].)

§6. From its definition in Section 2, $\sigma(a)$ is obviously an irrational number. We shall now decide whether this number is algebraic or not.

At this point it is convenient to introduce some abbreviations.

Put

$$u_n=\frac{q^{n\alpha(n)+n}-q^n+1}{(q^n-1)(q^{n\alpha(n)}-1)}-\frac{q^{(n+1)\alpha(n+1)+n}-q^n+1}{(q^{n+1}-1)(q^{(n+1)\alpha(n+1)}-1)}$$

and

$$E_n=\{\alpha(1)+2\alpha(2)q+\cdots+n\alpha(n)q^{n-1}\}(q-1),$$

and write $\sigma(a)$ as

$$\sigma(a)=\left(\frac{q^{\alpha(1)}}{(q-1)(q^{\alpha(1)}-1)}-\sum_{k=1}^{n-1}u_kq^{-E_k}\right)-\sum_{k=n}^{\infty}u_kq^{-E_k}.$$

Let now

$$D_n=(q-1)(q^2-1)\cdots(q^n-1)\,(q^{\alpha(1)}-1)(q^{2\alpha(2)}-1)\cdots(q^{n\alpha(n)}-1)\,,$$

and

$$B_n=D_nq^{E_{n-1}},\quad A_n=B_n\left(\frac{q^{\alpha(1)}}{(q-1)(q^{\alpha(1)}-1)}-\sum_{k=1}^{n-1}u_kq^{-E_k}\right),\quad R_n=\sum_{k=n}^{\infty}u_kq^{-E_k}\,.$$

Then $B_n>0$ and A_n are integers; R_n is a positive number, and

(3) $$\sigma(a)=\frac{A_n}{B_n}-R_n\,.$$

It follows easily from the definition of u_n that

$$\lim_{n\to\infty}u_n=\frac{q-1}{q}\,.$$

Since the numbers E_n increase sufficiently rapidly,

(4) $$R_n\sim(q-1)q^{-(E_n+1)}\,,$$

and n tends to infinity.

Further

$$D_n<q^{\{1+2+\cdots+n\}+\{\alpha(1)+2\alpha(2)+\cdots+n\alpha(n)\}}\,;$$

whence, by the definition of E_n,

(5) $$\lim_{n\to\infty}\frac{\log D_n}{E_n}=0\,.$$

§7. Next,

(6) $$\liminf_{n\to\infty}\frac{E_n}{E_{n-1}}\geqq q\,.$$

For let this assertion be false. There exists then a constant c satisfying

$$0<c<q$$

such that

$$E_n\leqq cE_{n-1}$$

for all sufficiently large n. But then, as n increases indefinitely,

$$E_n = O(c^n),$$

contrary to

$$E_n = \{\alpha(1) + 2\alpha(2)q + \cdots + n\alpha(n)q^{n-1}\}(q-1) \geqq n\alpha(n)(q^n - q^{n-1}).$$

Since $q \geqq 2$, relation (6) implies that there exists an infinite strictly increasing sequence of positive integers

$$N = \{n_1, n_2, n_3, \cdots\}$$

such that

(7) $$E_n > \tfrac{5}{3}E_{n-1} \quad \text{for} \quad n \in N.$$

Hence, by (3) and (4), for all sufficiently large $n \in N$,

$$0 < \left|\sigma(a) - \frac{A_n}{B_n}\right| < q^{-\frac{5}{3}E_{n-1}}.$$

Further, for such n, by the definition of B_n and by (5) and (7),

(8) $$B_n < q^{\frac{4}{3}E_{n-1}},$$

and therefore

$$0 < \left|\sigma(a) - \frac{A_n}{B_n}\right| < B_n^{-5/4}$$

if $n \in N$ is sufficiently large.

In this estimate, the denominator $B_n = D_n q^{E_{n-1}}$ tends to infinity and, by (5), (7) and (8),

$$D_n < B_n^{1/8}$$

for all sufficiently large $n \in N$. Moreover, the second factor $q^{E_{n-1}}$ has only finitely many bounded prime factors.

It follows then, by Ridout's generalization of Roth's theorem ([6], see also [3]), that $\sigma(a)$ is a transcendental number. We have thus proved the following result.

THEOREM 1. *If n runs over the positive integers, if, for every n, $\alpha(n)$ is a positive integer, and if in the definition of $\sigma(a)$,*

$$a_k = \alpha(n) \qquad \text{for} \qquad q^{n-1} \leqq k \leqq q^n - 1,$$

then $\sigma(a)$ is transcendental.

The following result can also be proved.

THEOREM 2. *Under the same hypothesis as in Theorem 1, $\sigma(a)$ is a Liouville number if and only if*

$$\sup_n \frac{E_n}{E_{n-1}} = \infty .$$

The proof is not difficult and may be omitted.

We have assumed that $\alpha(n)$ is always positive. Actually, this assumption is too restrictive, and Theorem 1 remains valid if $\alpha(n)$ is allowed to assume the value 0 provided there are infinitely many n for which $\alpha(n)$ is positive. Whenever $\alpha(n)$ vanishes, the representation (2) of $\sigma(a)$ becomes invalid. But, in order to obtain a correct representation of $\sigma(a)$, it suffices to omit in (2) the two contributions which correspond to each $\alpha(n) = 0$.

There remains the unsolved problem whether Theorem 1 has an analogue for general sequences a.

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