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On a Special Function

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Over 50 years ago, when I was his student at the University of Frankfurt a.M., C. L. Siegel explained to me how to apply Mellin's integral $e^{-t} = (1/2\pi i) \times \int \Gamma(s) t^{-s}\, ds$, where the integration is over a line parallel to the imaginary axis and to the right of $s = 0$, to the study of the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ in the neighborhood of roots of unity on the complex unit circle $|z| = 1$. I later could obtain similar results by means of Poisson's or Euler's summation formula. In the present note I return to this old problem and obtain estimates by means of a very elementary method. It has the further advantage that it allows the study of $f(z)$ in the neighborhood of points on the unit circle which are not roots of unity.

1. Let z be a complex variable. The power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

converges and defines a regular function when z lies in the unit disk

$$|z| < 1,$$

but it cannot be continued beyond this disk. For let

$$\epsilon = e^{2\pi i k/2^m},$$

where m and k are integers such that $m \geqslant 0$ and $0 \leqslant k \leqslant 2^m - 1$, be an arbitrary 2^mth root of unity. Then

$$f(z) = \sum_{n=0}^{m-1} (\epsilon r)^{2^n} + \sum_{n=m}^{\infty} r^{2^n}$$

if $z = \epsilon r$ and $0 \leqslant r < 1$, and here the first sum remains bounded while the second one tends to $+\infty$ as r tends to 1. Therefore all the 2^mth roots of unity

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are singular points of $f(z)$, and since these roots of unity are everywhere dense on the unit circle $|z| = 1$, this circle is a natural boundary for $f(z)$.

We shall now make this well-known result more precise by estimating how $f(z)$ behaves when z approaches the unit circle.

2. For this purpose write z in the form

$$z = e^{-t+\phi i},$$

where t is a positive number and ϕ a real number. We are interested in the behaviour of $f(z)$ as t, for arbitrary ϕ, tends to 0 and may therefore, without loss of generality, assume that already

$$0 < t \leqslant 1.$$

Let, as usual, $[x]$ denote the integral part of the real number x. Then associate with t the nonnegative integer

$$N = \left[\frac{\log(1/t)}{\log 2}\right]; \tag{1}$$

hence

$$2^N t \leqslant 1 < 2^{N+1} t. \tag{2}$$

The power series $f(z)$ can be split into the two sums

$$f(z) = f_1(z) + f_2(z),$$

where

$$f_1(z) = \sum_{n=0}^{N-1} z^{2^n} \quad \text{and} \quad f_2(z) = \sum_{n=N}^{\infty} z^{2^n}.$$

For the terms of $f_1(z)$,

$$z^{2^n} = e^{-2^n t} \cdot e^{2^n \phi i} = e^{2^n \phi i} + e^{2^n \phi i}(e^{-2^n t} - 1),$$

so that

$$|z^{2^n} - e^{2^n \phi i}| = 1 - e^{-2^n t}.$$

Now for real x,

$$e^x \geqslant 1 + x. \tag{3}$$

Therefore

$$1 - 2^n t \leqslant e^{-2^n t} \leqslant 1,$$

whence

$$0 \leqslant 1 - e^{-2^n t} \leqslant 2^n t.$$

It follows then from (2) and (3) that

$$\left| f_1(z) - \sum_{n=0}^{N-1} e^{2^n \phi i} \right| \leqslant \sum_{n=0}^{N-1} 2^n t = (2^N - 1)\, t \leqslant 1. \tag{4}$$

Next,

$$|f_2(z)| \leqslant \sum_{n=N}^{\infty} e^{-2^n t} \leqslant \sum_{k=1}^{\infty} e^{-2^N kt} = e^{-2^N t}(1 - e^{-2^N t})^{-1} = (e^{2^N t} - 1)^{-1},$$

where by (2) and (3),

$$e^{2^N t} - 1 \geqslant 2^N t \geqslant 1/2.$$

It follows that

$$|f_2(z)| \leqslant 2. \tag{5}$$

On combining the estimates (4) and (5), the following result is found.

THEOREM 1. *Let t and ϕ be real numbers where $0 < t \leqslant 1$, and let N be the nonnegative integer defined by* (1). *Then uniformly in t and ϕ,*

$$\left| f(z) - \sum_{n=0}^{N-1} e^{2^n \phi i} \right| \leqslant 3. \tag{6}$$

I have not tried to replace the constant 3 on the right-hand side by the best possible constant.

3. The definition (1) of N implies that

$$N \sim \frac{\log(1/t)}{\log 2},$$

and so it follows from (6) that

$$\frac{\log 2}{\log(1/t)} f(e^{-t+\phi i}) = \frac{1}{N} \sum_{n=0}^{N-1} e^{2^n \phi i} + O(1/N) \tag{7}$$

uniformly in t and ϕ if $0 < t \leqslant 1$.

This equation suggests the following notation. In general, as t tends to 0 through positive values, or equivalently, as N tends to infinity, neither the expression on the left-hand side of (7) nor the first term on the right-hand side of (7) needs tend to a unique limit. Therefore, for each fixed value of ϕ, denote by $S(\phi)$ the set of all possible limits of

$$\frac{\log 2}{\log(1/t)} f(e^{-t+\phi i})$$

as $t \to +0$, and similarly by $T(\phi)$ the set of all possible limits of

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{2^n \phi i}$$

as $N \to \infty$. The relation between t and N ensures then that always

$$S(\phi) = T(\phi). \tag{8}$$

However, exceptionally it may happen that the ordinary limit

$$\lim_{t \to +0} \frac{\log 2}{\log(1/t)} f(e^{-t+i\phi}), \qquad = s(\phi) \text{ say,}$$

or the ordinary limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2^n \phi i}, \qquad = t(\phi) \text{ say,}$$

does in fact exist. If this is so, then both limits exist simultaneously, and

$$s(\phi) = t(\phi). \tag{9}$$

The function $f(z)$ satisfies the functional equation

$$f(z) = f(z^2) + z.$$

From this it follows immediately that

$$S(2\phi) = S(\phi) \qquad \text{and} \qquad T(2\phi) = T(\phi), \tag{10}$$

and if $s(\phi)$ and $t(\phi)$ exist, also

$$s(2\phi) = s(\phi) \qquad \text{and} \qquad t(2\phi) = t(\phi). \tag{11}$$

In particular,

$$s(0) = t(0) = 1. \tag{12}$$

4. It is convenient to replace ϕ in the last formulas by $2\pi\psi$ where ψ is a further real number because the exponential function of ψ

$$e(\psi) = e^{2\pi i\psi}$$

has the period 1. Further put

$$S[\psi] = S(2\pi\psi), \quad T[\psi] = T(2\pi\psi), \quad s[\psi] = s(2\pi\psi), \quad t[\psi] = t(2\pi\psi),$$

so that always

$$S[\psi] = T[\psi],$$

and that

$$s[\psi] = t[\psi]$$

if these limits exist.

5. In the special case when ψ is a rational number, we can easily show that $t[\psi]$ and hence also $s[\psi]$ exist and determine their common value. Put

$$\psi = p/q,$$

where p and q are integers such that

$$0 \leqslant p \leqslant q - 1, \qquad (p, q) = 1.$$

If q is a power of 2, it follows from (11) that

$$t[p/q] = 1. \tag{13}$$

More generally, if $q = 2^k Q$ is the product of a power of 2 times an odd integer Q, by (11)

$$t[p/q] = t[p/Q]. \tag{14}$$

It suffices therefore to study the case when the denominator

$$q \text{ is odd.}$$

Denote by

$$r = \phi(q)$$

Euler's function of q, so that by Euler's theorem

$$2^r \equiv 1 \pmod{q},$$

hence

$$e(2^m p/q) = e(2^n p/q) \qquad \text{if} \quad m \equiv n \pmod{q}.$$

Hence, on writing the integer N as

$$N = Mr + m,$$

where M and m are integers such that

$$M \geqslant 0 \qquad \text{and} \qquad 0 \leqslant m \leqslant r - 1,$$

then

$$\sum_{n=0}^{N-1} e(2^n p/q) = M \sum_{n=0}^{r-1} e(2^n p/q) + \sum_{n=0}^{m-1} e(2^n p/q),$$

where we have used that $e(\psi)$ has period 1. In this formula the second sum has at most r terms and so its absolute value cannot exceed r. Further, as N tends to infinity, M/N has the limit i/r. It follows that $s[p/q]$ and $t[p/q]$ exist and are given by

$$s[p/q] = t[p/q] = \frac{1}{r} \sum_{n=0}^{r-1} e(2^n p/q), \tag{15}$$

where $r = \phi(q)$.

The finite sum on the right-hand side of this formula, when different from zero, is a Gaussian period from the theory of cyclotomy. (See Kummer [1] and Fuchs [2].)

6. When $\phi = 2\pi\psi$ is not a rational multiple of 2π, $s[\psi]$ and $t[\psi]$ need not exist. A simple example is given by the number

$$\psi = \sum_{n=1}^{\infty} d_n 2^{-n},$$

where the coefficients d_n are digits 0 and 1 defined as follows. First put $1! = 1$, digit $d_1 = 1$, then $2! = 2$ pairs of digits 0, 1 so that $d_2 = d_4 = 0$, $d_3 = d_5 = 1$.

Then put again 3! = 6 single digits 1, followed by 4! = 24 pairs of digits 0, 1. Generally, alternate between $(2n - 1)!$ single digits 1 and $(2n)!$ pairs of digits 0, 1. It is easily seen that the two sets $S[\psi] = T[\psi]$ contain at least two distinct limit points, hence that $s[\psi]$ and $t[\psi]$ do not exist with this choice of ψ.

In a different direction there is a classical theorem by Borel and Weyl which states that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(2^n \psi) = 0$$

for almost all real ψ. Hence by (7) for almost all points $e(\psi)$ on the unit circle for approach along the radius

$$f(e^{-t+2\pi i\psi}) = o(\log(1/t)).$$

In the neighborhood of the unit circle $f(z)$ oscillates violently as is clear from tabulating its values. The function has exactly one real zero $\neq 0$ at

$$-0.658\,626\,8,$$

and I found three pairs of complex roots

$$0.120\,314\,8 \pm i.0.934\,605\,9,$$
$$0.391\,862\,7 \pm i.0.898\,257\,6,$$
$$-0.685\,206\,2 \pm i.0.670\,534\,1.$$

It is highly probable that $f(z)$ has zeros in every neighborhood of the unit circle, but I have not proved this.

References

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