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On two definitions of the integral of a p-adic function

by

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In memory of Paul Turán

In his basic paper on functions of a p-adic variable Dieudonné [1], introduced a special kind of integral (primitive) of a continuous function. A completely different definition of such an integral was more recently given by M. van der Put (see A. C. M. van Rooij and W. H. Schikhof [2]). The aim of this note is to show that these two definitions lead to the same result. This is rather surprising because there is a large set of non-constant p-adic functions of derivative 0.

Since it simplifies the discussion, we shall study the two kinds of integrals for the class of functions $f: J \to Q_p$ where p is any positive rational prime, Q_p is the field of p-adic numbers, and $J = \{0, 1, 2, \ldots\}$ is the set of all non-negative rational integers. The set J is not closed, and its p-adic closure is the set $I = \{x \in Q_p;\ |x|_p \leqslant 1\}$ of all p-adic integers which is compact.

1. Let f: $J \to Q_p$ be an arbitrary function on J. The two integrals of f are defined by the following constructions.

Write $x \in J$ in the canonic form as

$$x = x_0 + x_1 p + x_2 p^2 + \ldots$$

where $x_0, x_1, x_2, \ldots$ are digits $0, 1, \ldots, p-1$. At most finitely many of these digits are distinct from 0; so, if $x \neq 0$, let $x_s \neq 0$ be the non-vanishing digit of largest suffix s. Firstly put

$$q(0) = 0, \qquad q(x) = x_s p^s \quad \text{for} \quad x \neq 0.$$

Secondly write

$$x^{(n)} = x_0 + x_1 p + \ldots + x_{n-1} p^{n-1} \qquad (n = 1, 2, 3, \ldots)$$

so that

$$x^{(n+1)} = x^{(n)} \quad \text{for} \quad n > s.$$

The Dieudonné integral of f is now defined by

$$D(x) = \sum_{n=1}^{\infty} (x^{(n+1)} - x^{(n)}) f(x^{(n)}).$$

Since the terms of this series vanish for $n > s$, there is no problem of convergence. One can show that, whenever f is continuous at a point x_0 of J, then $D'(x_0) = f(x)$, as required for an integral.

2. Let m be any integer in J. With m we associate a positive integer M where

$$M = 1 \quad \text{if} \quad m = 0,$$

while for $m \geqslant 1$ the integer M is chosen such that

$$p^{M-1} \leqslant m \leqslant p^M - 1.$$

Denote by $S(m)$ the ball consisting of all $x \in J$ for which

$$|x - m|_p \leqslant p^{-M},$$

and by $X(x, m)$ the characteristic function of $S(m)$ defined by

$$X(x, m) = \begin{cases} 1 & \text{if} \quad x \in S(m), \\ 0 & \text{otherwise.} \end{cases}$$

It can be proved that every function f: $J \to Q_p$ has a unique van der Put series

$$f(x) = \sum_{m=0}^{\infty} b_m X(x, m) \quad \text{for all } x \in J.$$

Here the coefficients b_m can be determined by the formulae

$$b_m = \begin{cases} f(m) & \text{if} \quad m = 0, 1, \ldots, p-1; \\ f(m) - f(m - q(m)) & \text{if} \quad m \geqslant p. \end{cases}$$

Since

$$X(x, m) = 0 \quad \text{if} \quad x < m,$$

the van der Put series for $f(x)$ breaks off after finitely many terms, and there is again no problem of convergence.

In the special case when $f(x)$ is the function x, we obtain the series

$$x = \sum_{m=0}^{\infty} q(m) X(x, m).$$

Once the van der Put series for $f(x)$ is known, its van der Put integral is defined by the development

$$P(x) = \sum_{m=0}^{\infty} b_m X(x, m)(x - m).$$

Also this integral satisfies the relation $P'(x_0) = f(x_0)$ at every point $x_0 \in J$ at which the function f is continuous, as it should be.

3. Without any restrictions on f we can now prove the following result.

THEOREM 1. *For every function* $f\colon J \to Q_p$,

$$D(x) = P(x) \quad \textit{for all } x \in J.$$

Proof. The van der Put series for $f(x)$ shows that it suffices to prove this theorem only for all the characteristic functions

$$f(x) = X(x, m).$$

Denote therefore by $D(x, m)$ and $P(x, m)$ the Dieudonné and the van der Put integrals of $X(x, m)$; we must prove that

$$D(x, m) = P(x, m) \quad \text{for all } x \in J.$$

This will be done by evaluating these two integrals explicitly, and we shall begin with the more difficult function $D(x, m)$.

Let x be an arbitrary element of J so that also $x^{(n)} \in J$ for all $n \geqslant 1$. If $X(x^{(n)}, m) = 0$ for all $n \geqslant 1$, then $D(x, m) = 0$; we exclude this easy case. There is then a smallest integer $N \geqslant 1$ such that $x^{(N)} \in S(m)$. Then

$$|x^{(N)} - m|_p \leqslant p^{-M}$$

and therefore there is a rational integer x^* such that

$$x^{(N)} = m + p^M x^*.$$

Here

$$p^{M-1} \leqslant m \leqslant p^M - 1,$$

from which it follows that x^* cannot be negative because then

$$x^{(N)} \leqslant m - p^M \leqslant -1,$$

contrary to $x^{(N)} \in J$. Therefore either

(1) $$x^{(N)} = m,$$

or

(2) $$x^{(N)} \geqslant m + p^M \geqslant p^M.$$

Now

$$\begin{aligned} x^{(N)} &= x_0 + x_1 p + \ldots + x_{N-1} p^{N-1} \\ &\leqslant (p-1) + (p-1)p + \ldots + (p-1)p^{N-1} \leqslant p^N - 1. \end{aligned}$$

Hence, in the case (2),

$$p^M \leqslant x^{(N)} \leqslant p^N - 1$$

and therefore $N \geqslant M+1$. It would then follow that

$$x^{(N)} = x^{(M)} + x_M p^M + \ldots + x_{N-1} p^{N-1},$$

and therefore

$$|x^{(N)} - x^{(M)}|_p \leqslant p^{-M},$$

whence also

$$|x^{(M)} - m|_p = |(x^{(M)} - x^{(N)}) + (x^{(N)} - m)|_p \leqslant p^{-M}.$$

Thus $x^{(M)} \in S(m)$, contrary to the minimum hypothesis for N.

Therefore the case (1) holds, and

(3) $$x^{(N)} = m.$$

We assert that moreover

(4) $$N = M.$$

For, if $N > M$, the proof just given leads to a contradiction; if, however, $N < M$, then

$$0 \leqslant x^{(N)} = m \leqslant p^N - 1 \leqslant p^{M-1} - 1 < p^{M-1},$$

and this likewise is false.

On account of (3) and (4) we can now prove that exactly

$$x^{(n)} \in S(m) \quad \text{for all } n \geqslant M.$$

For if $n \geqslant M+1$, we have again

$$x^{(n)} = x^{(M)} + x_M p^M + \ldots + x_{n-1} p^{n-1}$$

and therefore

$$|x^{(n)} - x^{(M)}|_p = |x^{(n)} - m|_p \leqslant p^{-M},$$

as asserted.

The integral $D(x, m)$ can now be determined and is found to have the value

$$D(x, m) = \begin{cases} \sum\limits_{n=M}^{\infty} (x^{(n+1)} - x^{(n)}) \times 1 = x - x^{(m)} = x - m & \text{if} \quad x \in S(m), \\ 0 & \text{otherwise.} \end{cases}$$

For $x^{(n)}$ becomes equal to x as soon as n is sufficiently large.

Since by definition also $P(x, m) = X(x, m)(x - m)$, we have proved the theorem.

4. From any integral of the arbitrary function $f(x)$ we obtain others by adding any function the derivative of which vanishes identically. In the present p-adic case there are very many such almost-constants. For instance, as C. S. Weisman has proved, every function

$$\sum_{m=0}^{\infty} \beta_m X(x, m),$$

where

$$\lim_{m\to\infty} m|\beta_m|_p = 0,$$

has everywhere the derivative 0.

Since there is then such a great choice of possible integrals of $f(x)$, the question may be asked whether the special integral $D(x) = P(x)$ has any distinguishing properties.

I obtained one such property. Write $f(x)$ and $P(x)$ as interpolation series

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \quad \text{and} \quad D(x) = \sum_{n=0}^{\infty} A_n \binom{x}{n}.$$

Then the coefficients A_n of the integral can be expressed as linear forms

$$(5) \qquad A_n = \sum_{m=0}^{n-1} c_{mn} a_m \quad (n \geqslant 1)$$

where the coefficients c_{mn} are *rational integers*. This is quite different from the position for functions of a real variable where, e.g.

$$\int \binom{x}{2} dx = \binom{x}{3} + \frac{1}{2}\cdot\binom{x}{2} - \frac{1}{12}\cdot\binom{x}{1} + \text{constant}$$

with *fractional* rational coefficients. In the p-adic case the Dieudonné–van der Put integral of $\binom{x}{2}$ is a rather more complicated infinite interpolation series

$$\sum_{n=1}^{\infty} c_{2n} \binom{x}{n}.$$

I shall establish and study the formulae (5) elsewhere.

References

[1] J. Dieudonné, *Sur les fonctions continues p-adiques*, Bull. Sci. Math. (2) 68 (1944), pp. 79–95.

[2] A. C. M. van Rooij and W. H. Schikhof, *Non-archimedean analysis*, Nieuw Archief v. Wiskunde (2), 19 (1971), pp. 120–160.

[3] C. S. Weisman, *On p-adic differentiability*, J. Number Theory 9 (1977), pp. 79–86.

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