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On a special nonlinear functional equation

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In memory of C. L. SIEGEL

We study analytic solutions of the functional equation

$$h(z)^2 - h(z^2) + c = 0, \tag{H}$$

where $c > 0$ is a parameter, and constant solutions are excluded. It suffices to consider solutions for which $h(z) - z^{-1}$ is regular in a neighbourhood of $z = 0$. If $0 < c \leqslant \frac{1}{4}$, $h(z)$ can be continued as a single-valued analytic function into the unit disc $|z| < 1$ where its only singularity is the pole at $z = 0$; the circle $|z| = 1$ is a natural boundary. On the other hand, if $c > \frac{1}{4}$, then by analytic continuation $h(z)$ becomes a multiple-valued function with an infinite sequence of quadratic branch points tending to every point of $|z| = 1$, and no branch of $h(z)$ can be continued beyond this circle.

A change of variable transforms (H) into the difference equation

$$g(Z+1) - g(Z) = g(Z)^2 + C,$$

where C is a real parameter. The solutions of this equation have properties similar to those of (H).

INTRODUCTION

While there are general methods for dealing with linear differential, difference, and functional equations, nonlinear problems of these kinds require special methods, and the solutions often depend rather discontinuously on the occurring parameters. This strange behaviour makes them of particular interest to the pure mathematician.

In the present paper I discuss a special nonlinear functional equation with just one parameter, and I establish the manner in which its main solution depends on this parameter.

1.

The functional equation to be considered is a special case of a more general equation studied by myself a few years ago.

Denote by p a prime and by

$$H(X, Y) = -(X^p - Y)(Y^p - X) + \sum_{m=0}^{p} \sum_{n=0}^{p} E_{mn} X^m Y^n \quad (E_{mn} = E_{nm})$$

a symmetric polynomial in X and Y with complex coefficients E_{mn}, where

$$E_{pp} = E_{p-1,p} = E_{p,p-1} = 0.$$

As I have proved (Mahler 1976), there exists a unique formal Laurent series

$$h(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n$$

with complex coefficients such that

$$H(h(z^p), h(z)) = 0.$$

This formal series $h(z)$ has in fact a positive radius of convergence, r say, and it defines for $|z| < r$ an analytic function with a simple pole at $z = 0$.

If $r \geqslant 1$, $h(z)$ exists thus as a single-valued function in $|z| < 1$. If, however, $r < 1$, then by means of the functional equation $h(z)$ can still be continued into $|z| < 1$, but now possesses infinitely many algebraic branch points in this disc tending to every point of $|z| = 1$, and $h(z)$ is multiple-valued in the unit disc. The dependence of the radius r on the coefficients E_{mn} is still an unsolved problem.

I have solved this problem for the special equation considered in this paper. When reducible, $\mathrm{H}(X, Y)$ splits into a product of the form

$$\mathrm{H}(X, Y) = -\left(X^p + \sum_{n=0}^{p-2} c_n X^n - Y\right)\left(Y^p + \sum_{n=0}^{p-2} c_n Y^n - X\right),$$

with certain complex coefficients $c_0, c_1, \ldots, c_{p-2}$, and the functional equation for $h(z)$ takes the simpler form

$$h(z)^p + \sum_{n=0}^{p-2} c_n h(z)^n - h(z^p) = 0.$$

We shall be dealing here with the case when $p = 2$ and $c_0 = c$, so that

$$h(z)^2 - h(z^2) + c = 0. \tag{H}$$

To simplify the considerations, it will be assumed that the parameter c is positive, but it is probably possible to deal similarly with the case when c is an arbitrary complex number. When $c = 0$, the only solutions $h(z)$ analytic in a neighbourhood of $z = 0$ are integral powers of z, and for $c \neq 0$ the only solutions regular in a neighbourhood of this point are constants. It is also clear that if $h(z)$ satisfies (H), then so does the function $h(z^m)$ where m is arbitrary. Non-trivial solutions $h(z)$ of (H) necessarily have a singular point at $z = 0$. By the general result quoted at the beginning of § 1 this singular point may be assumed to be a simple pole of residue 1.

2.

Let then for the moment

$$h(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n \tag{1}$$

be any formal Laurent series with complex coefficients satisfying the functional equation (H). It follows immediately from (H) that

$$h(-z)^2 = h(z^2) - c = h(z)^2,$$

so that $h(z)$ is either an even or an odd function. From the series (1), it is necessarily odd:

$$h(-z) = -h(z). \tag{2}$$

Hence all the coefficients b_n with even n vanish, and we may write (1) as

$$h(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^{2n-1}, \tag{3}$$

where $a_n = -b_{2n-1}$.

On substituting the formal series (3) into (H), it follows that

$$z^{-2} - 2z^{-1} \sum_{n=1}^{\infty} a_n z^{2n-1} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n z^{2(m+n-1)} - z^{-2} + \sum_{n=1}^{\infty} a_n z^{2(2n-1)} + c = 0.$$

Since in this identity the sum of the coefficients of each occurring power of z must be zero, we deduce the system of recursive formulae:

$$a_1 = \tfrac{1}{2}c, \quad a_{2n} = \sum_{j=1}^{n-1} a_j a_{2n-j} + \tfrac{1}{2}(a_n^2 + a_n), \quad a_{2n+1} = \sum_{j=1}^{n} a_j a_{2n-j+1} \quad (n = 1, 2, 3, \ldots). \tag{4}$$

For the lowest values of n,

$$a_1 = \tfrac{1}{2}c, \quad a_2 = \tfrac{1}{8}(c^2 + 2c), \quad a_3 = \tfrac{1}{16}(c^3 + 2c^2), \quad a_4 = \tfrac{1}{128}(5c^4 + 10c^3 + 9c^2 + 18c).$$

Generally a_n is a non-constant polynomial $a_n(c)$ in c with non-negative rational coefficients. Thus, for $c > 0$, *all the coefficients a_n are positive-valued increasing functions of c.*

It is convenient to put

$$g(z) = \sum_{n=1}^{\infty} a_n z^{2n-1}$$

so that

$$h(z) = z^{-1} - g(z).$$

By Mahler (1976), $g(z)$ has a positive radius of convergence, $r = r(c)$ say. For all z satisfying $|z| < r$,

$$|g(z)| \leqslant g(|z|),$$

because the coefficients a_n are positive. An upper estimate for $g(|z|)$ can be obtained as follows.

If, first, $c \geqslant 2$, the recursive formulae (4) show that

$$a_n \geqslant 1 \quad \text{and therefore} \quad \tfrac{1}{2}(a_n^2 + a_n) \leqslant a_n^2 \quad (n = 1, 2, 3, \ldots).$$

Therefore

$$a_1 = \tfrac{1}{2}c \quad \text{and} \quad a_n \leqslant \sum_{j=1}^{n-1} a_j a_{n-j} \quad (n = 2, 3, 4, \ldots).$$

Hence if new coefficients A_n are defined by the recursive formulae

$$A_1 = \tfrac{1}{2}c \quad \text{and} \quad A_n = \sum_{j=1}^{n-1} A_j A_{n-j} \quad (n = 2, 3, 4, \ldots),$$

then

$$0 \leqslant a_n \leqslant A_n \quad (n = 1, 2, 3, \ldots).$$

Put now

$$G(t) = \sum_{n=1}^{\infty} A_n t^n,$$

so that

$$G(t)^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_m A_n t^{m+n} = \sum_{n=2}^{\infty} (A_1 A_{n-1} + A_2 A_{n-2} + \ldots + A_{n-1} A_1) t^n$$

$$= \sum_{n=2}^{\infty} A_n t^n = G(t) - \tfrac{1}{2}ct,$$

whence

$$G(t)^2 - G(t) + \tfrac{1}{2}ct = 0, \quad G(t) = \tfrac{1}{2}\{1 \pm (1-2ct)^{\frac{1}{2}}\}.$$

By its power series, $G(t)$ vanishes at $t = 0$. Therefore the negative sign of the square root applies, and

$$G(t) = \tfrac{1}{2}\{1 - (1-2ct)^{\frac{1}{2}}\}.$$

It follows that the power series for $G(t)$ converges in the interval

$$0 \leqslant t \leqslant 1/2c.$$

In this interval,

$$\frac{1}{G(t)} = \left\{\frac{1-(1-2ct)^{\frac{1}{2}}}{2}\right\}^{-1} = \frac{1+(1-2ct)^{\frac{1}{2}}}{ct} \geqslant \frac{1}{ct}.$$

Hence

$$0 \leqslant G(t) \leqslant ct \quad \text{if} \quad c \geqslant 2 \quad \text{and} \quad 0 \leqslant t \leqslant 1/2c.$$

Now

$$|g(z)| \leqslant g(|z|) = \sum_{n=1}^{\infty} a_n |z|^{2n-1} \leqslant \sum_{n=1}^{\infty} A_n |z|^{2n-1} = G(|z|^2)/|z|.$$

The estimate for $G(t)$ implies therefore that

$$|g(z)| \leqslant g(|z|) \leqslant c|z|^2/|z| = c|z| \quad \text{if} \quad c \geqslant 2 \quad \text{and} \quad |z| < (2c)^{-\frac{1}{2}}.$$

Next, since the coefficients a_n are strictly increasing functions of c, the function $g(|z|)$ is for fixed $|z|$ a strictly increasing function of $c > 0$. Hence the estimate for $|g(z)|$ just proved shows that also

$$|g(z)| \leqslant 2|z| \quad \text{if} \quad 0 < c \leqslant 2 \quad \text{and} \quad |z| < \tfrac{1}{2}.$$

On combining these upper bounds, we arrive at the following result.

Theorem 1. *Let c be positive and z complex. Then*

$$|h(z) - z^{-1}| \leqslant \begin{cases} c|z| & \text{if} \quad c \geqslant 2 \quad \text{and} \quad |z| < (2c)^{-\frac{1}{2}}, \\ 2|z| & \text{if} \quad 0 < c \leqslant 2 \quad \text{and} \quad |z| < \tfrac{1}{2}. \end{cases} \tag{5}$$

We shall later apply this theorem to the computation of $h(z)$.

3.

From the Laurent series (3) for $h(z)$, the derivative of $h(z)$ is

$$h'(z) = -z^{-2} - \sum_{n=1}^{\infty} a_n(2n-1)\, z^{2n-2},$$

which converges and is negative on the interval $0 < z < r$ on the real axis. As z runs over this interval from 0 to r, $h(z)$ strictly decreases from $+\infty$ at $z = 0$ to a certain limit at $z = r$ that is either a finite real number or is equal to $-\infty$.

As a consequence, we can prove the following result.

THEOREM 2. *If* $c > \frac{1}{4}$, *then* $0 < r < 1$.

Proof. We know already that r is positive; we have thus only to prove that $r < 1$. Assume on the contrary that $r \geqslant 1$. Then the limit

$$h(1) = \lim_{\substack{z \to 1-0 \\ z \text{ real}}} h(z)$$

exists and is either a finite real number, or is equal to $-\infty$. Now, from (H),

$$h(1)^2 - h(1) + c = 0,$$

whence

$$h(1) = \tfrac{1}{2}\{1 \pm (1-4c)^{\frac{1}{2}}\}.$$

But both these values are non-real, which gives a contradiction.

Theorem 2 shows that

$$0 < r^2 < r. \tag{6}$$

4.

Before continuing with the study of $h(z)$ for $c > \frac{1}{4}$, let us investigate a special sequence $\{p_n(w)\}$ of polynomials $p_n(w)$ that is defined by

$$p_1(w) = w^2 + c, \quad p_{n+1}(w) = p_n(w)^2 + c \quad (n = 1, 2, 3, \ldots). \tag{7}$$

These polynomials are of degree 2^n in w; they have constant coefficients and highest coefficient unity. Moreover, they satisfy the following further identities,

$$p_{n+1}(w) = p_n(w^2 + c) \quad (n = 1, 2, 3, \ldots). \tag{8}$$

For, from the definition,

$$p_2(w) = (w^2+c)^2 + c = p_1(w^2+c),$$

and if for any $n \geqslant 2$ it has already been proved that

$$p_n(w) = p_{n-1}(w)^2 + c = p_{n-1}(w^2+c),$$

then also

$$p_{n+1}(w) = p_n(w)^2 + c = p_{n-1}(w^2+c)^2 + c = p_n(w^2+c),$$

which proves that (8) is true for all positive integers n.

On differentiating the equations (7),

$$p_1'(w) = 2w,$$
$$p_2'(w) = 2p_1(w)\,p_1'(w),$$
$$p_3'(w) = 2p_2(w)\,p_2'(w),$$
$$\vdots$$
$$p_n'(w) = 2p_{n-1}(w)\,p_{n-1}'(w),$$

and therefore

$$p_n'(w) = 2^n w p_1(w)\,p_2(w)\ldots p_{n-1}(w) \quad (n = 1, 2, 3, \ldots); \tag{9}$$

here there are no p-factors for $n = 1$.

Define now a sequence $\{C_n\}$ of constants C_n by

$$C_0 = 0, \quad C_n = \pm(C_{n-1} - c)^{\frac{1}{2}}; \quad \text{hence} \quad C_{n-1} = c + C_n^2 \quad (n = 0, 1, 2, \ldots). \tag{10}$$

Thus C_n has for $n \geqslant 1$ not one but 2^n allowed values since its definition involves n square roots. The constants C_n are in fact the zeros of the polynomials $p_n(z)$.

The polynomial $p_1'(w)$ vanishes at $w = 0 = C_0$, the polynomial $p_2'(w)$ both at $w = C_0$ and at $w = \pm(-c)^{\frac{1}{2}} = C_1$. For general n it is easily deduced from (9) that

$$p_n'(w) \quad \text{has the zeros} \quad C_0, C_1, \ldots, C_{n-1}. \tag{11}$$

5.

The functional equation (H) for $h(z)$ is equivalent to

$$h(z^2) = p_1(h(z)).$$

On squaring z repeatedly and applying the equations (H) and (8), induction on n leads to the more general functional equation

$$h(z^{2^n}) = p_n(h(z)) \quad (n = 1, 2, 3, \ldots), \tag{12}$$

because

$$h(z^{2^n}) = p_{n-1}(h(z^2)) = p_{n-1}(h(z)^2 + c) = p_n(h(z)).$$

For the application of (12), denote by R an arbitrary number in the interval $0 < R < 1$ and by n any positive integer such that

$$R^{2^n} < r.$$

In the region

$$U_R: \quad 0 < |z| \leqslant R,$$

the function $h(z^{2^n})$ certainly is regular. By the functional equation (12) we have defined $h(z)$ as an algebraic function of $h(z^{2^n})$ which evidently has no poles in U_R, but may have a finite number of algebraic branch points. Moreover, by the theory of algebraic functions, $h(z)$ must at these branch points satisfy the further condition

$$p_n'(h(z)) = 0. \tag{13}$$

It follows then from § 4 that at these branch points $h(z)$ has one of the values

$$h(z) = C_0, C_1, \ldots, C_{n-1},$$

and hence, by (H), $h(z^2)$ has one of the corresponding values

$$h(z^2) = c, C_0, C_1, \ldots, C_{n-2}.$$

Allow now R to tend to one, and hence n to tend to infinity. The region U_R then becomes

$$U: \quad 0 < |z| < 1,$$

and we conclude that the only singularities of $h(z)$ in U are possibly infinitely many algebraic branch points. Moreover, at these branch points $h(z)$ may assume only the values

$$h(z) = C_0, C_1, C_2, \ldots,$$

and $h(z^2)$ the corresponding values

$$h(z^2) = c, C_0, C_1, \ldots .$$

6.

So far, we have not yet proved that the analytic continuation of $h(z)$ into U does in fact have branch points. We first construct a subset $\mathscr{R}_0$ of U in which the analytic continuation of $h(z)$ is still regular.

Denote by m any positive integer and by ρ, for $m = 1$, either of the two numbers

$$\rho = \pm r,$$

but for $m \geqslant 2$ let ρ denote any one of the 2^{m-1} complex roots of the equation

$$\rho^{2^{m-1}} = -r.$$

With each pair (m, ρ) associate the line segment $L(m, \rho)$ of all points

$$z = t\rho \quad \text{where} \quad 1 \leqslant t < |\rho|^{-1}.$$

All these line segments are subsets of U. We denote by

$$\mathscr{R}_0 = U \setminus \bigcup_{(m,\rho)} L(m, \rho)$$

the set of all points in U that do *not* belong to any line segment $L(m, \rho)$.

The set $\mathscr{R}_0$ has the property that if z is one of its points, then also z^2 lies in $\mathscr{R}_0$.

This assertion is obvious when z is not of the form $z = t\rho$ for any pair (m, ρ) and any real number t with $0 < t < 1$ because $|z| > |z^2|$. If, however, z is of this form for a certain pair (m, ρ) and a certain real number t, then either $m = 1$ and $\rho = \pm r$, and hence $|z| < r$ and therefore also $|z^2| < r$ so that certainly $z^2 \in \mathscr{R}_0$; or $m \geqslant 2$, when $z^2 = t^2\rho^2$ where $0 < t^2 < 1$, and ρ^2 evidently is a root of the equation belonging to $m' = m - 1$, whence again $z^2 \in \mathscr{R}_0$.

The following property can now be proved.

THEOREM 3. *Let $c > \frac{1}{4}$. If $h(z)$, as defined by the Laurent series* (3), *is continued analytically by means of the functional equation* (H) *into $\mathscr{R}_0$, then it is regular at all points of this set.*

Proof. Let the assertion be false so that there exists at least one singular point, z_0 say, in $\mathscr{R}_0$. The points $z_0, z_0^2, z_0^4, z_0^8, \ldots$ all belong to $\mathscr{R}_0$, tend to 0, and finally lie in the set $0 < |z| \leqslant \frac{1}{2}r$ in which $h(z)$ by its series (3) is regular. Hence there exists a positive integer n such that $z_0^{2^{n-1}}$ is singular, but $z_0^{2^n}$ is regular; again write z_0 for $z_0^{2^{n-1}}$, so that z_0, but *not* z_0^2, is a singular point of $h(z)$ in $\mathscr{R}_0$.

This implies that the function $h(z^2)$ is regular in a neighbourhood of $z = z_0$. In this neighbourhood, by (H),

$$h(z) = \{h(z^2) - c\}^{\frac{1}{2}}. \tag{14}$$

Therefore $h(z)$ can only then be singular at $z = z_0$ if this point is a *quadratic branch point* of $h(z)$ and if, moreover,

$$h(z_0) = 0 \quad \text{and} \quad h(z_0^2) = c.$$

(Note that this corresponds to the first possibility for algebraic branch points of $h(z)$ considered in § 5.)

The two properties of $h(z)$, first that $z = z_0$ is a quadratic branch point, and secondly that $h(z_0) = 0$, have the consequence that there must pass through the point $z = z_0$ two continuous differentiable curve arcs, Γ_1 and Γ_2 say, that (i) intersect at this point at right angles, and (ii) are both mapped by the function $w = h(z)$ onto two intervals on the *real axis* in the w-plane that both contain $w = 0$ as interior points.

By (i), the straight line through the two points $z = 0$ and $z = z_0$ cannot be a tangent to both arcs Γ_1 and Γ_2 at the point z_0. Choose the notation such that the line is not a tangent to Γ_1 at this point. Hence if the points z on Γ_1 are written in the form

$$z = u\mathrm{e}^{2\pi \mathrm{i} v},$$

where u and v are real, and $0 < u < 1$, then v is not a constant as z tends to z_0 on Γ_1, but is a *variable* continuous function of z.

It has been proved in the theory of diophantine approximations that the set of all real numbers v for which the inequality

$$|v - (p/q)| < q^{-3}$$

has infinitely many solutions in integers $p, q > 0$ is of Lebesgue measure 0. This implies that there exists some point

$$z^* = u^* \mathrm{e}^{2\pi \mathrm{i} v^*} \quad (u^*, v^* \text{ real}, 0 < u^* < 1)$$

on Γ_1 arbitrarily close to the point $z = z_0$ such that

$$|v^* - (p/q)| \geqslant q^{*-3}$$

for all integers p and all sufficiently large integers $q > 0$. It follows that

$$|2^n v^* - \text{nearest integer}| \geqslant 2^{-2n}$$

and therefore

$$|\sin(2^n 2\pi v^*)| \geqslant 2^{-3n}$$

for all sufficiently large positive integers n.

By this construction, the point $z = z^*$ lies on Γ_1, and hence $h(z^*)$ is a *real* number. Since $h(z^2) = h(z)^2 + c$, it follows that all the function values

$$h(z^{*2^n}) \quad (n = 0, 1, 2, \ldots)$$

are real. On the other hand, by the Laurent series (3),

$$h(z^{*2^n}) = (z^{*2^n})^{-1} - \sum_{k=1}^{\infty} a_k (z^{*2^n})^{2k-1} = z^{*-2^n} + O(u^{*2^n}),$$

and both z^{*2^n} and u^{*2^n} tend to 0 as n tends to infinity. Hence the imaginary part of $h(z^{*2^n})$ is equal to

$$-u^{*-2^n} \sin(2^n 2\pi v^*) + O(u^{*2^n}),$$

so that for large n its absolute value is not less than

$$\tfrac{1}{2} u^{*-2^n} 2^{-3n}$$

and so tends to infinity with n. But this means that $h(z^{*2^n})$ is not a real number when n is sufficiently large, contrary to the construction.

Corollary. *At the non-real points z of $\mathscr{R}_0$ the imaginary parts of z and $h(z)$ have opposite signs.*

Proof. This assertion is implicit in the last proof. By it, $h(z)$ can be real only for real z (although, in fact, it need not be real for such points). Hence the imaginary part of $h(z)$ has fixed signs for both $\operatorname{Im} z > 0$ and for $\operatorname{Im} z < 0$. The sign of $\operatorname{Im} h(z)$ can be determined by considering points z of small modulus and applying once again the series (3) for $h(z)$.

7.

We have not yet proved that $h(z)$ does in fact have branch points in the larger set U. The proof will be based on the following lemma.

Lemma 1. *Let $\zeta \in U$, so that $\zeta \neq 0$. Denote by ζ_0 either root of the equation $\zeta_0^2 = \zeta$. Let further*

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n (z-\zeta)^{\frac{1}{2}n}$$

be a series in powers of $(z-\zeta)^{\frac{1}{2}}$ that converges in a neighbourhood of $z = \zeta$, and in which the coefficient ϕ_0 is not a real positive number and the coefficient ϕ_1 does not vanish. Then

$$\psi(z) = \{\phi(z^2) - c\}^{\frac{1}{2}}$$

can in a neighbourhood of $z = \zeta_0$ *be written as a convergent series*

$$\psi(z) = \sum_{n=0}^{\infty} \psi_n (z-\zeta_0)^{\frac{1}{2}n}$$

in powers of $(z-\zeta_0)^{\frac{1}{2}}$, *and here the coefficient* ψ_0 *is neither* 0 *nor a positive number, and the coefficient* ψ_1 *is distinct from* 0.

Proof. The identity

$$z^2-\zeta = (z-\zeta_0)(z+\zeta_0) = 2\zeta_0(z-\zeta_0)+(z-\zeta_0)^2$$

shows that $|z^2-\zeta|$ becomes smaller than any given positive number if $|z-\zeta_0|$ is sufficiently small. Hence the series

$$\phi(z^2)-c = (\phi_0-c)+\sum_{n=1}^{\infty}\phi_n(z^2-\zeta)^{\frac{1}{2}n}$$

converges in a neighbourhood of $z = \zeta_0$ and can here be rearranged into a convergent series

$$\phi(z^2)-c = (\phi_0-c)+\sum_{n=1}^{\infty}\phi_n^*(z-\zeta_0)^{\frac{1}{2}n}$$

in powers of $(z-\zeta_0)^{\frac{1}{2}}$. Since $c > \frac{1}{4}$ by hypothesis, the coefficient ϕ_0-c neither vanishes nor is a positive real number, and the coefficient $\phi_1^* = \phi_1(2\zeta_0)^{\frac{1}{2}}$ does not vanish. It follows also that the square root $\psi_0 = (\phi_0-c)^{\frac{1}{2}}$ is neither 0 nor positive real.

Now, by the definitions of $\psi(z)$ and of ψ_0,

$$\psi(z) = \left\{\psi_0^2+\sum_{n=1}^{\infty}\phi_n^*(z-\zeta_0)^{\frac{1}{2}n}\right\}^{\frac{1}{2}} = \psi_0\left\{1+\psi_0^{-2}\sum_{n=1}^{\infty}\phi_n^*(z-\zeta_0)^{\frac{1}{2}n}\right\}^{\frac{1}{2}}.$$

Hence, by the binomial theorem, $\psi(z)$ can in a certain neighbourhood of $z = \zeta_0$ be expanded into a convergent series

$$\psi(z) = \psi_0\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\psi_0^{-2k}\left\{\sum_{n=1}^{\infty}\phi_n^*(z-\zeta_0)^{\frac{1}{2}n}\right\}^k = \sum_{n=0}^{\infty}\psi_n(z-\zeta_0)^{\frac{1}{2}n}$$

in powers of $(z-\zeta_0)^{\frac{1}{2}}$ with certain coefficients ψ_k. We know already that ψ_0 has the required properties, and evidently $\psi_1 = \frac{1}{2}\psi_0^{-2}\phi_1^*$ does not vanish.

Remark. The assumption that ϕ_0 is not positive, and hence that $\psi_0 \neq 0$, has the consequence that $\psi(z)$ cannot have a branch point of higher order than 2 at $z = \zeta_0$, while the assumption that $\phi_1 \neq 0$ implies that both $\phi(z)$ at $z = \zeta$, and $\psi(z)$ at $z = \zeta_0$ do have branch points of order 2.

8.

The continuation of $h(z)$ into $\mathscr{R}_0$ is not only regular, but also single-valued because $\mathscr{R}_0$ evidently is simply connected. It will now be proved that $h(z)$ has infinitely many branch points in the topological closure $\overline{U}$ of $\mathscr{R}_0$.

These singular points lie already on the sets $L(m,\rho)$, and we begin with a study of the special line segment $L(1,r)$ which is identical with the interval $[r,1)$ on the positive real axis.

That $z=r$ is a singular point of $h(z)$ follows from the following two facts. (i) The function $h(z)$ necessarily has at least one singular point on the circle of convergence $|z|=r$ of the Laurent series (3). (ii) Since all the coefficients a_n in (3) have the same (positive) sign, the point $z=r$ which belongs to $L(1,r)$ must be singular. On the other hand, $h(z)$ is regular at, and hence in a neighbourhood of, the point $z=r^2$ which by $r^2<r$ lies in $\mathscr{R}_0$.

By the same considerations as in the proof of theorem 3, $z=r$ is a quadratic branch point of $h(z)$, and

$$h(r)=0, \quad h(r^2)=c.$$

We note also that, by § 3, $h(z)$ is positive, real and strictly decreasing on the interval $(0,r]$ on the real axis, decreasing from large positive values near $z=0$ to 0 at $z=r$.

The behaviour of $h(z)$ near the branch point $z=r$ can be studied by means of a series in powers of $(z-r)^{\frac{1}{2}}$. The function $h(z^2)$ is regular in a neighbourhood of $z=r$ and here can be written as a convergent Taylor series

$$h(z^2)=\sum_{n=0}^{\infty} c_n(z-r)^n,$$

where the coefficients c_n are defined by

$$c_0=c \quad \text{and} \quad n!c_n=(\mathrm{d}/\mathrm{d}z)^n\, h(z^2)|_{z=r} \quad (n=1,2,3,\ldots).$$

All these coefficients are real numbers. In particular,

$$c_1=\mathrm{d}h(z^2)/\mathrm{d}z|_{z=r}=2rh'(r^2),$$

so that by § 3 the factor $h'(r^2)$ and hence also the coefficient c_1 are negative. Let γ_0 be the positive square root of $-c_1$ and fix the square root $c_1^{\frac{1}{2}}$ by

$$c_1^{\frac{1}{2}}=-\mathrm{i}\gamma_0.$$

The reason for the choice of the minus sign will soon become clear.

In a neighbourhood of $z=r$, just as in the proof of lemma 1,

$$h(z)=\{h(z^2)-c\}^{\frac{1}{2}}=-\mathrm{i}\gamma_0(z-r)^{\frac{1}{2}}\left\{1+\sum_{n=2}^{\infty} c_1^{-1}c_n(z-r)^{n-1}\right\}^{\frac{1}{2}}$$

$$=-\mathrm{i}\gamma_0(z-r)^{\frac{1}{2}}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\sum_{n=2}^{\infty} c_1^{-1}c_n(z-r)^{n-1}\right)^k,$$

and therefore

$$h(z)=-\mathrm{i}(z-r)^{\frac{1}{2}}\sum_{n=0}^{\infty}\gamma_n(z-r)^n, \tag{15}$$

where the coefficients γ_n are real numbers, and γ_0 has already been defined and does not vanish.

This representation shows that $h(z)$ has a quadratic branch point at $z = r$ and therefore is at least two-valued. Hence the Riemann surface of $h(z)$, $\mathscr{R}$ say, has at least two sheets. One of these sheets is $\overline{\mathscr{R}}_0$; call the other sheet $\overline{\mathscr{R}}_1$. These two sheets hang together at the branch point $z = r$ and intersect each other, say along a subinterval of $L(1, r)$.

From now on denote by $h(z)$ the analytic continuation of the function defined by the series (3) onto $\mathscr{R}$, and by $h_0(z)$ and $h_1(z)$ the branches of $h(z)$ on the sheets $\overline{\mathscr{R}}_0$ and $\overline{\mathscr{R}}_1$, respectively. A similar notation with suffixes will be used for other branches of $h(z)$.

The square root $(z-r)^{\frac{1}{2}}$ in (15) has not yet been fixed. Let us assume that $-i(z-r)^{\frac{1}{2}} = (r-z)^{\frac{1}{2}}$ is taken positive real on the line segment $(0, r)$ on the positive real axis of $\overline{\mathscr{R}}_0$; from there continue the square root analytically into the sheets $\overline{\mathscr{R}}_0$ and $\overline{\mathscr{R}}_1$ of $\mathscr{R}$. With this choice of the sign of the square root, the development (15) represents the branch $h_0(z)$ of $h(z)$, while the other sign gives the branch $h_1(z)$. Hence

$$h_1(z) = -h_0(z). \tag{16}$$

Furthermore, since $h(z)$ is an odd function of z, also the point $z = -r$ is a quadratic branch point of $h(z)$ on $\overline{\mathscr{R}}_0$. It follows from (15) that $h(z)$ has in a neighbourhood of this point the development

$$h(z) = -(z+r)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \gamma_n (z+r)^n, \tag{17}$$

with the same coefficients γ_n as in (15). Choose here the square root $-(z+r)^{\frac{1}{2}}$ such that it is negative real on the interval $(-r, 1)$ of the negative real axis of $\overline{\mathscr{R}}_0$, and continue it from here analytically onto the Riemann surface $\mathscr{R}$. It is evident that the branch of $h(z)$ so defined coincides with $h_0(z)$ on $\overline{\mathscr{R}}_0$. The other branch is equal to $-h_0(z)$ and hence, by (16), is the branch $h_1(z)$ on $\overline{\mathscr{R}}_1$. The two sheets $\overline{\mathscr{R}}_0$ and $\overline{\mathscr{R}}_1$ are thus connected at both quadratic branch points $z = r$ and $z = -r$.

9.

A series

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n (z-\zeta)^{\frac{1}{2}n}$$

in powers of $(z-\zeta)^{\frac{1}{2}}$ is said to be *admissible at* $z = \zeta$ if it has the following properties.

(*a*) The series converges in a neighbourhood of $z = \zeta$;

(*b*) $0 < |\zeta| < 1$;

(*c*) the coefficient ϕ_0 is not a positive real number, and the coefficient ϕ_1 does not vanish.

Denote by $\zeta_0 = \zeta^{\frac{1}{2}}$ either of the two square roots of ζ so that again

$$0 < |\zeta_0| < 1.$$

Lemma 1 then states that the new series

$$\psi(z) = \{\phi(z^2) - c\}^{\frac{1}{2}} = \sum_{n=0}^{\infty} \psi_n (z - \zeta_0)^{\frac{1}{2}n}$$

in powers of $(z-\zeta_0)^{\frac{1}{2}}$ is admissible at $z = \zeta_0$. We may thus replace $\phi(z)$ by $\psi(z)$ and repeat the construction.

This procedure will now be applied to the branch $h_0(z)$ of $h(z)$. The series (15) for $h_0(z)$ evidently is admissible at $z = r$; furthermore, by (H),

$$h_0(z) = \{h_0(z^2) - c\}^{\frac{1}{2}}$$

if the correct sign of the square root is taken. A repeated application of lemma 1 shows thus that $h_0(z)$ *is admissible at all points of the sequence*

$$\Sigma(1, r) = \{r, r^{\frac{1}{2}}, r^{\frac{1}{4}}, r^{\frac{1}{8}}, \ldots\}$$

which are therefore quadratic branch points of $h_0(z)$. Since $h_0(z)$ is an odd function of z, we may immediately add that $h_0(z)$ *is also admissible at all points of the sequence*

$$\Sigma(1, -r) = \{-r, -r^{\frac{1}{2}}, -r^{\frac{1}{4}}, -r^{\frac{1}{8}}, \ldots\},$$

so that also these points are quadratic branch points of $h_0(z)$. In fact, much more can be proved. Define the pairs (m, ρ) as in § 6, and put

$$\rho = \epsilon|\rho|,$$

so that ϵ is a $2m$th root of unity. Associate with (m, ρ) the sequence of points

$$\Sigma(m, \rho) = \{\epsilon|\rho|, \epsilon|\rho|^{\frac{1}{2}}, \epsilon|\rho|^{\frac{1}{4}}, \epsilon|\rho|^{\frac{1}{8}}, \ldots\}$$

which lie on the line segment $L(m, \rho)$ and tend to the limit ϵ on the unit circle.

Theorem 4. *Let* $c > \frac{1}{4}$, *and let* (m, ρ) *run over all the pairs defined in* § 6. *Then* $h_0(z)$ *is admissible at all points of the corresponding sequence* $\Sigma(m, \rho)$ *and hence all these points are quadratic branch points of* $h_0(z)$.

Proof. The assertion has already been proved for the two special sequences $\Sigma(1, r)$ and $\Sigma(1, -r)$ that belong to $m = 1$. Now let $m \geqslant 2$ and assume that the theorem is already known to be true for the values $1, 2, \ldots, m-1$ and all the corresponding roots ρ. If now (m, ρ) is an allowed pair, let $\epsilon|\rho|^{2^{-n}}$ be any element of $\Sigma(m, \rho)$; here n is some non-negative integer. Then also $(m-1, \rho^2)$ is an allowed pair, and $(\epsilon|\rho|^{2^{-n}})^2$ evidently belongs to $\Sigma(m-1, \rho^2)$. Therefore, by the induction hypothesis, $h_0(z)$ is admissible at the point $(\epsilon|\rho|^{2^{-n}})^2$, and it follows from lemma 1 that it is also admissible at $z = \epsilon|\rho|^{2^{-n}}$. This proves the assertion.

Corollary. *The quadratic branch points of* $h_0(z)$ *tend to every point on the unit circle* $|z| = 1$. *This circle consists thus entirely of singular points of* $h_0(z)$ *and hence is a natural boundary of both* $h_0(z)$ *and* $h_1(z)$.

10.

Theorem 3 established that $h_0(z)$ has no singular points in the set $\mathscr{R}_0$. The frontier of $\mathscr{R}_0$ consists of the origin $z = 0$, the unit circle, and of all the infinitely many line segments $L(m, \rho)$. Here the origin is a pole of $h_0(z)$, the unit circle is a natural boundary, and, by theorem 4, the sequence $\Sigma(m, \rho)$ on each line segment $L(m, \rho)$ consists of quadratic branch point of $h_0(z)$. On the other hand, the following result holds.

THEOREM 5. *Let $c > \frac{1}{4}$. There does not exist any singular point of $h_0(z)$ on a line segment $L(m, \rho)$ that is not contained in $\Sigma(m, \rho)$.*

Proof. Let the assertion be false, and let z_0 be a singular point of $h_0(z)$ on $L(m, \rho)$ that does not belong to $\Sigma(m, \rho)$. Then z_0 has the form

$$z_0 = \epsilon |\rho|^{2^{-t}},$$

where, as in § 9, $\epsilon = \rho/|\rho|$, and t is a certain positive number that is not an integer. Here the number ρ was defined by

$$\rho = \pm r$$

if $m = 1$, and was for $m \geqslant 2$ one of the 2^{m-1} roots of the equation

$$\rho^{2^{m-1}} = -r.$$

Hence in either case,

$$|\rho|^{2^{m-1}} = r \quad \text{and} \quad |\rho|^{2^m} = r^2.$$

Consider now the sequence of successive squares

$$\{z_0, z_0^2, z_0^4, z_0^8, \ldots\}.$$

This sequence has the limit 0; hence its elements finally lie in $\mathscr{R}_0$ and then, by theorem 3, are regular points of $h_0(z)$. It follows that there is a smallest non-negative integer n such that $z_0^{2^n}$ is a singular point and $z_0^{2^{n+1}}$ is a regular point of $h_0(z)$. By the same considerations as in the proof of theorem 3 this implies that

$$h_0(z_0^{2^n}) = 0 \quad \text{and} \quad h_0(z_0^{2^{n+1}}) = c.$$

Therefore, by the corollary to theorem 3, the singular point $z_0^{2^n}$, $= z_1$ say, lies on the real axis and its square $z_0^{2^{n+1}} = z_1^2$ lies on the positive real axis. Further

$$h_0(z_1) = 0 \quad \text{and} \quad h_0(z_1^2) = c. \tag{18}$$

Since $h_0(z)$ is an odd function, also $-z_1$ is a singular point of $h_0(z)$. There is then no loss of generality in assuming that z_1 itself is positive; for otherwise replace z_1 by $-z_1$. This does not affect the equations (18).

Next

$$z_1 = \pm z_0^{2^n} = \pm \epsilon^{2^n} |\rho|^{2^{n-t}} = r^{2^{n-m-t+1}}$$

since $\pm\epsilon^{2^n} = +1$ by our choice of the sign of z_1.

Put

$$T = -(n-m-t+1), \quad \text{so that} \quad z_1 = r^{2^{-T}}.$$

Since t is not an integer, the same is true for T. Moreover, T is positive because by theorem 3 the singular point z_1 cannot be less than r.

Denote by $N = [T]$ the integral part of T so that

$$N \geqslant 0 \quad \text{and} \quad N < T < N+1.$$

Hence

$$r^{2^{-N}} < z_1 < r^{2^{-(N+1)}},$$

and therefore

$$r^2 < z_1^{2^{N+1}} < r. \tag{19}$$

We combine finally the second equation (18) with the functional equation (H) written in the form

$$h_0(z^2) = h_0(z)^2 + c.$$

It follows that

$$c = h_0(z_1^2) \leqslant h_0(z_1^4) \leqslant \ldots \leqslant h_0(z_1^{2^{N+1}}).$$

However, as z runs from r^2 to r along the positive real axis, $h_0(z)$ decreases steadily from the value c at $z = r^2$ to the value 0 at $z = r$, and so, by (19), the function value $h_0(z_0^{2^{N+1}})$ cannot be greater or equal to c. This contradiction proves the assertion.

11.

More information about the multiple-valued function $h(z)$ is contained in the following result.

THEOREM 6. *Let $c > \frac{1}{4}$. The function $h(z)$ is infinitely-many-valued, and its Riemann surface $\mathscr{R}$ has infinitely many sheets.*

Proof. We shall study $h(z)$ on small closed curves with centres at the branch points in $\Sigma(1, r)$ and find that the number of branches at the successive points of this sequence tends to infinity.

Denote by t a very small positive constant. The branch $h_0(z)$ of $h(z)$ is regular at the point $z = r^2$ and at all points of the circle

$$C: |z - r^2| = t$$

of centre r^2 and radius t. Here the points on C may be written in the form

$$z = r^2 + t e^{\theta i}, \quad \text{where} \quad 0 \leqslant \theta \leqslant 2\pi.$$

As θ runs from 0 to 2π, C is described in the positive direction.

When z lies on C and n is any non-negative integer,

$$z^{2^{-n-1}} = (r^2 + t e^{\theta i})^{2^{-n-1}} = r^{2^{-n}}(1 + r^{-2} t e^{\theta i})^{2^{-n-1}}.$$

Denote here by $r^{2^{-n}}$ the *positive* real value of the root, and define the factor

$$(1+r^2 t e^{\theta i})^{2^{-n-1}}$$

by means of the binomial expansion as

$$\sum_{k=0}^{\infty} \binom{2^{-n-1}}{k} (r^{-2} t e^{\theta i})^k = 1 + 2^{-n-1} r^{-2} t e^{\theta i} + O(t^2).$$

It follows that when z runs over the circle C, $z^{2^{-n-1}}$ describes a closed curve, $C(n+1)$ say, with points given by

$$C(n+1): \quad z = r^{2^{-n}} + 2^{-n-1} r^{-2+2^{-n}} t e^{\theta i} + O(t^2).$$

Again, as θ runs from 0 to 2π, $C(n+1)$ is described in the positive direction. The curve $C(n+1)$ is approximately a circle of centre $r^{2^{-n}}$ and radius $2^{-n-1} r^{-2+2^{-n}} t$. It is also clear that when z^2 runs over C or over $C(n+1)$, z runs over $C(1)$ or over $C(n+2)$, respectively.

By the Laurent series (3), $h_0(z)$ has at the points z of C the form

$$h_0(z) = c + \gamma t e^{\theta i} + O(t^2), \tag{20}$$

where

$$\gamma = h_0'(r^2)$$

is a negative real number. This estimate is uniform in θ.

First let $z^2 = r^2 + t e^{\theta i}$ run over C, and hence $z = r + 2^{-1} r^{-1} t e^{\theta i} + O(t^2)$ over $C(1)$. By (20) and by the functional equation (H), two branches of $h(z)$ have on $C(1)$ the values

$$h(z) = \pm \{h_0(z) - c\}^{\frac{1}{2}} = \pm \{\gamma t e^{\theta i} + O(t^2)\}^{\frac{1}{2}} = \pm (\gamma t)^{\frac{1}{2}} \{e^{\theta i} + O(t)\}^{\frac{1}{2}},$$

which differ only by a factor -1. On choosing the sign of $(\gamma t)^{\frac{1}{2}}$ suitably, this leads to the equations

$$h_0(z) = -(\gamma t)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}}) \quad \text{and} \quad h_1(z) = +(\gamma t)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}}), \tag{21}$$

which agrees with the earlier formulae (15) and (17) when z lies on $C(1)$. The exponential factor $e^{\frac{1}{2}\theta i}$ and the different signs show again that $h_0(z)$ and $h_1(z)$ are connected at the quadratic branch point $z = r$.

Next let $z^2 = r + 2^{-1} r^{-1} t e^{\theta i} + O(t^2)$ run over $C(1)$, and hence $z = r^{\frac{1}{2}} + 2^{-2} r^{-\frac{3}{2}} t e^{\theta i} + O(t^2)$ over $C(2)$. On applying (H) to representations (21) of $h_0(z)$ and $h_1(z)$ on $C(1)$, we obtain now four branches of $h(z)$ on $C(2)$, given by

$$\pm \{h_0(z) - c\}^{\frac{1}{2}} = \pm \{-c - (\gamma t)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}})\}^{\frac{1}{2}},$$

and

$$\pm \{h_1(z) - c\}^{\frac{1}{2}} = \pm \{-c + (\gamma t)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}})\}^{\frac{1}{2}},$$

respectively. This gives after a slight simplification,

$$\pm \{(-c)^{\frac{1}{2}} + (\gamma t/c)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}})\}$$

and

$$\pm \{(-c)^{\frac{1}{2}} - (\gamma t/c)^{\frac{1}{2}} e^{\frac{1}{2}\theta i} + O(t^{\frac{3}{2}})\},$$

where the factor $(\gamma t/c)^{\frac{1}{2}}$ may still have either sign. If this sign is chosen correctly, then on $C(2)$

$$h_0(z) = (-c)^{\frac{1}{2}} + (\gamma t/c)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\theta \mathrm{i}} + O(t^{\frac{3}{2}}) \tag{22}$$

and therefore

$$h_1(z) = -h_0(z) = -(-c)^{\frac{1}{2}} - (\gamma t/c)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\theta \mathrm{i}} + O(t^{\frac{3}{2}}). \tag{23}$$

In addition, there are two new branches, $h_2(z)$ and $h_3(z)$ say, of $h(z)$:

$$h_2(z) = (-c)^{\frac{1}{2}} - (\gamma t/c)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\theta \mathrm{i}} + O(t^{\frac{3}{2}}), \tag{24}$$

and

$$h_3(z) = -(-c)^{\frac{1}{2}} + (\gamma t/c)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\theta \mathrm{i}} + O(t^{\frac{3}{2}}). \tag{25}$$

Hence

$$h_3(z) = -h_2(z).$$

Both pairs of branches $h_0(z)$, $h_1(z)$ and $h_2(z)$, $h_3(z)$ of $h(z)$ have separate quadratic branch points at $z = r^{\frac{1}{2}}$, and thus the Riemann surface $\mathscr{R}$ of $h(z)$ has four sheets $\mathscr{R}_0, \mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ which are connected in pairs $(\mathscr{R}_0, \mathscr{R}_1)$ and $(\mathscr{R}_2, \mathscr{R}_3)$ at two separate quadratic branch points $z = r^{\frac{1}{2}}$ which lie on top of each other.

Since, for example,

$$h_0(z)^2 = h_0(z^2) - c \quad \text{and} \quad h_2(z) = -h_0(z^2) - c,$$

it also follows that

$$h_0(z)^2 + h_2(z)^2 + 2c = 0, \tag{26}$$

identically in z. Three further identities of the same kind arise if in this formula $h_0(z)^2$ is replaced by $h_1(z)^2$, or $h_2(z)^2$ by $h_3(z)^2$, or both.

This procedure can be repeated, with the result that at each step the number of branches of $h(z)$ and the number of sheets of $\mathscr{R}$ is doubled. At the next step let $z^2 = r^{\frac{1}{2}} + 2^{-2}r^{-\frac{3}{2}}t\mathrm{e}^{\theta \mathrm{i}} + O(t^2)$ run over $C(2)$, hence $z = r^{\frac{1}{4}} + 2^{-3}r^{-\frac{7}{4}}t\mathrm{e}^{\theta \mathrm{i}} + O(t^2)$ over $C(3)$. We now obtain four quadratic branch points at $z = r^{\frac{1}{4}}$ which lie one above the other and which correspond to the pairs of branches

$$\pm\{h_j(z^2) - c\}^{\frac{1}{2}} \quad (j = 0, 1, 2, 3).$$

This leads on $C(3)$ to eight branches

$$h_j(z) \quad (j = 0, 1, 2, \ldots, 7)$$

of $h(z)$, with the corresponding sheets $\mathscr{R}_0, \mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_7$ of $\mathscr{R}$. The new branches come again in pairs that differ only by a factor -1 and are connected by the corresponding branch point over $z = r^{\frac{1}{4}}$. An old branch like $h_0(z)$ and a new branch like $h_4(z)$ are connected by an algebraic equation

$$h_0(z)^4 + h_4(z)^4 + 2c\{h_0(z)^2 + h_4(z)^2\} + (2c^2 + 2c) = 0.$$

Continuing thus, we obtain an infinite sequence of branches of $h(z)$, where any two of these are connected by an algebraic equation with constant coefficients.

It would be of great interest to elucidate the complete structure of the Riemann surface $\mathscr{R}$ of $h(z)$ and to find uniformizing functions. I conjecture that there exist uniformizing functions $F_1(Z)$, $F_2(Z)$ with the following properties.

(*a*) *Both* $F_1(Z)$ *and* $F_2(Z)$ *are regular and single-valued on the open disc*

$$D: |Z| < 1.$$

(*b*) *The equations*

$$z = F_1(Z), \quad h(z) = F_2(Z)$$

establish a one-to-one correspondence between the points of the Riemann surface $\mathscr{R}$ *and the points of* D.

(*c*) *Each of the functions* $F_1(Z)$ *and* $F_2(Z)$ *satisfies a functional equation of the form*

$$\Psi_j(F_j(Z^{2k}), F_j(Z), Z) = 0 \quad (j = 1, 2),$$

where Ψ_1 *and* Ψ_2 *are polynomials with constant coefficients, and* k *is a certain positive integer.*

This section for the case when $c > \frac{1}{4}$ is concluded with the statement of a problem. We saw that the functional equation (H) has for $c > \frac{1}{4}$ only multi-valued solutions $h(z)$ if one branch $h_0(z)$ is assumed to have a simple pole at $z = 0$.

Problem. Does (H) *have a single valued regular solution in* $0 < |z| < 1$ *that is not a constant and has an essential singularity at* $z = 0$?

12.

From now on we deal with the analytic solutions $h(z)$ of (H) given in a neighbourhood of $z = 0$ by the Laurent series (3), but assume that the parameter c lies in the interval $0 < c \leqslant \frac{1}{4}$. We shall find that the radius of convergence r of the series (3) is now equal to $r = 1$, and hence that $h(z)$ is regular and single-valued for $0 < |z| < 1$.

For this purpose we try to develop $h(z)$ into a power series in powers of the parameter c, of the form

$$h(z) = z^{-1} - \sum_{n=1}^{\infty} c^n h_n(z), \tag{27}$$

where the factors $h_n(z)$ are regular functions of z inside the unit disc $D: |z| < 1$ that are independent of c.

Since $h(z)$ is an odd function of z, it follows that

$$-z^{-1} - \sum_{n=1}^{\infty} c^n h_n(-z) = -\left\{z^{-1} - \sum_{n=1}^{\infty} c^n h_n(z)\right\},$$

whence, on comparing the coefficients of c^n on both sides of this identity,

$$h_n(-z) = -h_n(z) \quad (n = 1, 2, 3, \ldots).$$

It follows in particular that

$$h_n(0) = 0 \quad (n = 1, 2, 3, \ldots). \tag{28}$$

Next, the functional equation (H) is equivalent to the identity

$$\left\{z^{-2}-2z^{-1}\sum_{n=1}^{\infty} c^n h_n(z)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c^{m+n} h_m(z)\,h_n(z)\right\}-z^{-2}+\sum_{n=1}^{\infty} c^n h_n(z^2)+c = 0.$$

Here the coefficients of the different powers of c must add to give zero. Therefore, on putting

$$s_n(z) = \sum_{k=1}^{n-1} h_k(z)\,h_{n-k}(z) \quad (n = 2, 3, 4, \ldots), \tag{29}$$

the following recursive system of functional equations is obtained:

$$h_1(z) = \tfrac{1}{2}z+\tfrac{1}{2}zh_1(z^2),$$

and

$$h_n(z) = \tfrac{1}{2}zs_n(z)+\tfrac{1}{2}zh_n(z^2) \quad (n = 2, 3, 4, \ldots).$$

These equations may be applied repeatedly, with z replaced successively by z^2, z^4, z^8, ..., and lead to the following expansions:

$$h_1(z) = \frac{z}{2}+\frac{z}{2}\frac{z^2}{2}+\frac{z}{2}\frac{z^2}{2}\frac{z^4}{2}+\ldots = z^{-1}\sum_{k=1}^{\infty} 2^{-k}z^{2^k},$$

and for $n = 2, 3, 4, \ldots$

$$h_n(z) = \frac{z}{2}s_n(z)+\frac{z}{2}\frac{z^2}{2}s_n(z^2)+\frac{z}{2}\frac{z^2}{2}\frac{z^4}{2}s_n(z^4)+\ldots = z^{-1}\sum_{k=1}^{\infty} 2^{-k}z^{2^k}s_n(z^{2^k}).$$

It is obvious that for positive real z all the functions $h_n(z)$ and $s_n(z)$ are positive, real and strictly increasing. Moreover, all the series are still convergent for $z = 1$. For, trivially,

$$h_1(1) = \sum_{k=1}^{\infty} 2^{-k} = 1,$$

from which it follows easily by induction on n that the values $s_n(1)$ exist and that

$$h_n(1) = \sum_{k=1}^{\infty} 2^{-k}s_n(1) = s_n(1) \quad (k = 2, 3, 4, \ldots).$$

Further the numbers $s_n(1)$ satisfy the recursive formulae

$$s_1(1) = 1, \quad s_n(1) = \sum_{k=1}^{n-1} s_k(1)\,s_{n-k}(1) \quad (n = 2, 3, 4, \ldots).$$

To evaluate the numbers $s_n(1)$, form the generating series

$$V(t) = \sum_{n=1}^{\infty} s_n(1)\,t^n,$$

where t is an indeterminate. From $s_1(1) = 1$ and by the recursive formulae for $s_n(1)$,

$$V(t)^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} s_m(1)\,s_n(1)\,t^{m+n} = \sum_{n=2}^{\infty} t^n\sum_{k=1}^{n-1} s_k(1)\,s_{n-k}(1) = \sum_{n=2}^{\infty} s_n(1)\,t^n = V(t)-t,$$

so that $V(t)$ satisfies the quadratic equation $V(t)^2 - V(t) + t = 0$ and is equal to

$$V(t) = \tfrac{1}{2}\{1 \pm (1-4t)^{\frac{1}{2}}\}.$$

Since $V(0) = 0$, the minus sign is the correct one, and by the binomial theorem,

$$V(t) = \tfrac{1}{2}\{1-(1-4t)^{\frac{1}{2}}\} = \tfrac{1}{2}\left\{1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n t^n\right\}.$$

Therefore

$$h_n(1) = s_n(1) = (-1)^{n-1} \binom{\frac{1}{2}}{n} 2^{2n-1} = 2^{n-1} \frac{1 \times 3 \times 5 \times \ldots \times (2n-3)}{n!}. \tag{30}$$

Here for $n = 1$ the right-hand side must be replaced by 1.

It is clear that all the series $h_n(z)$ converge absolutely and uniformly on the closed unit disc $\bar{D}: |z| \leqslant 1$ and here satisfy the inequalities

$$|h_n(z)| \leqslant h_n(1) \quad (n = 1, 2, 3, \ldots). \tag{31}$$

Hence the functions $h_n(z)$ are regular on D and continuous on the larger set $\bar{D}$.

13.

The assertion at the beginning of § 12 can now be proved.

THEOREM 7. *Let $0 < c \leqslant \frac{1}{4}$. Then $h(z)$, as defined by the Laurent series (3), is regular and single-valued in the region $0 < |z| < 1$ and is continuous on the unit circle $|z| = 1$.*

Proof. For $0 < c \leqslant \frac{1}{4}$ and $|z| \leqslant 1$, by (30) and (31),

$$\sum_{n=1}^{\infty} c^n |h_n(z)| \leqslant \sum_{n=1}^{\infty} (\tfrac{1}{4})^n |h_n(1)| = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \binom{\frac{1}{2}}{n}.$$

Here the series on the right-hand side is known to be convergent and in fact to have the sum $\frac{1}{2}$. The development (27) for $h(z)$ is therefore uniformly convergent in the closed unit disc $|z| \leqslant 1$ if the polar term z^{-1} is omitted. From this the assertion follows immediately.

We still have to prove that the radius of convergence r of (3) is equal to unity and not greater than unity. This assertion is contained in the following result.

THEOREM 8. *Let $0 < c \leqslant \frac{1}{4}$. Then the unit circle $|z| = 1$ is a natural boundary of $h(z)$.*

Proof. Since, by (3) and (27),

$$h(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^{2n-1} = z^{-1} - \sum_{n=1}^{\infty} c^n h_n(z),$$

identically in c and z,

$$\sum_{n=1}^{\infty} a_n z^{2n-1} = \sum_{n=1}^{\infty} c^n h_n(z).$$

If in this identity the coefficients a_n are replaced by their expressions as polynomials in c, and the functions $h_n(z)$ by their power series in z, then the two sides of this formulae become power series in c and z with terms that are identical except for their order; and all terms of the power series are positive when $0 < c \leqslant \frac{1}{4}$ and $0 < z \leqslant 1$. We know already that the series $\Sigma_{n=1}^{\infty} c^n h_n(1)$ converges; the same is therefore true for the series $\Sigma_{n=1}^{\infty} a_n$. By (H),

$$h(1)^2 - h(1) + c = 0;$$

hence $h(1) = \frac{1}{2}\{1+(1-4c)^{\frac{1}{2}}\}$. Here the square root must be taken with the plus sign because, as c tends to 0, the coefficients a_n tend to 0 and hence $h(1)$ tends to 1.

Since then $h(1)$ is positive, in the range $0 < c \leqslant \frac{1}{4}$,

$$0 < \sum_{n=1}^{\infty} a_n < 1.$$

Let now θ be any real number. Then

$$h(e^{\theta i}) = e^{-\theta i} - \sum_{n=1}^{\infty} a_n e^{(2n-1)\theta i},$$

and here

$$|e^{-\theta i}| = 1, \quad \left|\sum_{n=1}^{\infty} a_n e^{(2n-1)\theta i}\right| \leqslant \sum_{n=1}^{\infty} a_n < 1.$$

It follows that

$$|h(e^{\theta i})| \geqslant 1 - \sum_{n=1}^{\infty} a_n > 0,$$

and hence that $h(z)$ *does not vanish anywhere on the unit circle* $|z| = 1$.

Next, on differentiating the functional equation (H),

$$h(z)\,h'(z) - zh'(z^2) = 0. \tag{32}$$

On putting $z = 1$, this gives $\{h(1)-1\}\,h'(1) = 0$. Hence $h'(1) = 0$ because $h(1) < 1$. Since $h(z)$ is odd, naturally also $h'(-1) = 0$.

Let now n be a positive integer and ϵ any 2^nth root of unity, and hence ϵ^2 a 2^{n-1}st root of unity. We know already that $h'(\epsilon) = 0$ if $\epsilon^2 = 1$. Let us assume that for some $n \geqslant 2$ it has been shown that $h'(\eta) = 0$ if $\eta^{2^{n-1}} = 1$. If now $\epsilon^{2^n} = 1$, by (32),

$$h(\epsilon)\,h'(\epsilon) - \epsilon h'(\epsilon^2) = 0.$$

Here $h(\epsilon) \neq 0$, while by the induction hypothesis $h'(\epsilon^2) = 0$. Therefore also $h'(\epsilon) = 0$. Thus $h'(z)$ *vanishes at all* 2^nth *roots of unity where* $n = 1, 2, 3, \ldots$.

These 2^nth roots of unity tend to every point $e^{\theta i}$ on the unit circle. Thus if $h(z)$ and so also $h'(z)$ were regular in a neighbourhood of $z = e^{\theta i}$, then $h'(z)$ would be identically zero, and $h(z)$ would be a constant. Since this is false, there are no regular points on the unit circle, which proves the assertion.

One can show that not only $h'(z)$, but all derivatives $h'(z), h''(z), h(z), \ldots$ vanish at all 2^nth roots of unity.

14.

The previous results will now be applied to the solutions of a special nonlinear difference equation.

Denote by δ an arbitrarily small constant in the interval $0 < \delta < \frac{1}{2}\pi$, and by $Z = X + Yi$, where X and Y are real numbers, a new complex variable. This variable will be restricted to the horizontal strip S in the Z-plane defined by

$$S: \quad -(\tfrac{1}{2}\pi - \delta) \leqslant Y \ln 2 \leqslant +(\tfrac{1}{2}\pi - \delta),$$

so that

$$\sin\delta \leqslant \cos(Y \ln 2) \leqslant 1.$$

Let the original variable z be connected with Z by the equation

$$z = e^{-2^Z} = e^{-2^X\{\cos(Y\ln 2) + i\sin(Y\ln 2)\}}.$$

Hence

$$|z| = e^{-2^X \cos(Y\ln 2)} \leqslant e^{-2^X \sin\delta}.$$

It follows that when Z lies in S, $|z|$ tends to 1 when X tends to $-\infty$ and to 0 when X tends to $+\infty$. For all Z in S the point z satisfies $0 < |z| < 1$.

Let now $h(z)$ be again the solution of (H) considered in the previous sections. We define an analytic function $g(Z)$ for $Z \in S$ by the equation

$$g(Z) = h(e^{-2^Z}) - \tfrac{1}{2}.$$

A trivial calculation shows that $g(Z)$ satisfies the difference equation

$$g(Z+1) - g(Z) = g(Z)^2 + C, \quad \text{where} \quad C = c - \tfrac{1}{4}. \tag{G}$$

Since c was assumed positive, only results for $C > -\frac{1}{4}$ are obtained, when we transfer the properties proved for $h(z)$ to the new function.

By the Laurent series (3) of $h(z)$, we can assert that the difference equation (G) has a unique analytic solution $g(Z)$ such that for sufficiently large X

$$g(Z) = e^{2^Z} - \frac{1}{2} - \sum_{n=1}^{\infty} a_n e^{-(2n-1)2^Z},$$

where the coefficients a_n are the same as in § 2. This function $g(Z)$ is regular and single-valued in S when $-\frac{1}{4} < C \leqslant 0$, but it is infinitely many-valued if $C > 0$, and then has an infinite sequence of quadratic branch points in S.

15.

Consider again the function $h(z)$. It is not difficult to compute this function, both for real z in the interval $0 < z < r$ where r as before is the radius of convergence of the Laurent series (3), and for complex z.

Let us begin with the case when z is real and lies in the interval $0 < z < r$. The point $z = r$ may be excluded because, as we have learned,

$$h(r) = 0 \quad \text{if} \quad c > \tfrac{1}{4}, \quad \text{and} \quad r = 1, \quad h(1) = \tfrac{1}{2}\{1 + (1-4c)^{\frac{1}{2}}\} \quad \text{if} \quad 0 < c \leqslant \tfrac{1}{4}.$$

Thus, in either case, $0 < z < 1$. There exists then a smallest positive integer n such that

$$\max(c, 2)\, z^{2^n} < 10^{-13}. \tag{33}$$

(The upper bound 10^{-13} is chosen because the calculator works to this precision.) Then

$$z^{2^n} < c^{-1}10^{-13} < (1/2c)^{\frac{1}{2}} \quad \text{and} \quad z^{2^n} < 2^{-1}10^{-13} < \tfrac{1}{2}.$$

Hence it follows from theorem 1 that

$$|h(z^{2^n}) - z^{-2^n}| \leqslant \max(c, 2)\, z^{2^n} < 10^{-13}.$$

Thus, with an error less than 10^{-13},

$$h(z^{2^n}) = z^{-2^n}.$$

Having found $h(z^{2^n})$, we finally obtain $h(z)$ by applying (H) n times in the form

$$h(z) = \{h(z^2) - c\}^{\frac{1}{2}},$$

with all the square roots taken with the positive sign.

For such computations I used a Texas Instruments TI-59 programmable calculator with a PC-100A printer. The program is as follows,

LBL A RCL 00 x^2 STO 00 R/S

LBL B RCL 00 $1/x$ STO 00 R/S

LBL C (RCL 00 − RCL 01) $\sqrt{x}$ STO 00 R/S.

One first stores the values of z and c in the registers R_{00} and R_{01}, respectively, then presses key A n times until the displayed number z^{2^n} satisfies the inequality (33). Next one presses key B once and key C n times. The displayed number is $h(z)$.

By means of this program $h(z)$ has been tabulated for certain values of the parameter c. Naturally if $c > \frac{1}{4}$, the program does not work for $z > r$ because then $h(z)$ is not real.

For $c > \frac{1}{4}$ the same program allows one to evaluate the radius of convergence r. This is best done by using the equation $h(r^2) = c$. One may apply Newton's method, or simply trial and error. (See table 1.)

16.

To calculate $h(z)$ for $z > r$, or more generally for complex z, a different method must be used. If, first, $|z|$ is not too large compared with r, say $|z| \leqslant 0.4$ if $c = 2$ and $|z| \leqslant 0.5$ if $c = \frac{1}{4}$, about 10 terms of the Laurent series (3) give $h(z)$ with sufficient precision, once the Laurent coefficients $a_1, a_2, \ldots, a_9$ have been calculated, which can easily be done from the recursive formulae in § 2, by using the calculator.

Since $h(z)$ is an odd function and has real Laurent coefficients, it obviously suffices to consider variables z in the first quadrant of the z-plane. Write z in the trigonometrical form $z = Re^{\phi i}$ where R is positive, and for convenience ϕ is measured in degrees, thus lying between 0° and 90°.

The program for the evaluation of $h(z)$ is now as follows.

LBL A STO 10 PRT x^2 STO 11 R/S STO 12 PRT $\times$ 2 = STO 13 a_1 STO 51 a_2

STO 52 a_3 STO 53 a_4 STO 54 a_5 STO 55 a_6 STO 56 a_7 STO 57 a_8 STO 58

a_9 STO 59 51 STO 40 D

LBL B (RCL 10 $1/x$ $\times$ RCL 12 COS) $x \Leftrightarrow t$ (RCL 10 $1/x$ RCL 12 SIN) + / − Σ^+

INV SBR

LBL C (RCL 10 $\times$ RCL 12 COS $\times$ RCL IND 40) + / − $x \Leftrightarrow t$ (RCL 10 $\times$ RCL 12 SIN

$\times$ RCL IND 40) + / − Σ^+ 1 SUM 40 RCL 11 PROD 10 RCL 13 SUM 12 INV SBR

LBL D B C C C C C C C C C RCL 04 PRT RCL 01 PRT CMS CLR $x \Leftrightarrow t$ CLR R/S

To apply this program, key in the value of R, press A, then key in the value of ϕ in degrees and press R/S. After about a minute the printer will print out the values of the real and the imaginary parts of $h(z)$.

For larger values of R the Laurent series cannot be used and we then apply the functional equation (H), possibly several times. It is necessary to remember that in the first quadrant the real part of $h(z)$ is positive and the imaginary part negative.

Provided that the Master Library is installed in the calculator, we can obtain $h(z)$ from $h(z^2)$ by the following program.

PRG 05 [key in the real part of $h(z^2)$] − 2 = A [key in the imaginary part of $h(z^2)$] A D

The calculator then prints the real part of $h(z)$, and on pressing the key $x \Leftrightarrow t$ it prints the imaginary part of $h(z)$. It may be necessary to correct the signs of either or both. If R is close to 1, this procedure may have to be repeated several times.

Table 1. Radii of convergence

c	r	r^2
100	0.0997515	0.0099504
10	0.3087500	0.0953266
5	0.4269910	0.1823213
2	0.6345845	0.4026975
1	0.8157256	0.6654082
0.5	0.9639731	0.9292442
0.4	0.9890169	0.9782732
0.3	0.9998002	0.9996004
0.25	1.0000000	1.0000000

Reference

Mahler, K. 1976 On a class of nonlinear functional equations connected with modular functions. *J. Aust. math. Soc.* A **22**, 65–118.