Reprinted from JOURNAL OF NUMBER THEORY
Vol. 17, No. 3, December 1983

On a Theorem in the Geometry of Numbers in a Space of Laurent Series

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Communicated by the Editor

Received March 3, 1982

Proof of a general inequality connecting point sets with lattices in a space of Laurent series.

In 1948 Chabauty [1] and Rogers [2] simultaneously and independently proved the inequality

$$\Delta(S) \prod_{k=1}^{n} m_k(S, \Lambda) \leqslant 2^{(n-1)/2} d(\Lambda)$$

for the successive minima $m_k(S, \Lambda)$ of an arbitrary point set S relative to any lattice Λ in real n-dimensional space R^n. Both Chabauty [1] and Mahler [3] proved independently that the constant factor $2^{(n-1)/2}$ is best possible.

We shall now establish the analogous inequality

$$\Delta(S) \prod_{k=1}^{n} m_k(S, \Lambda) \leqslant 2^n d(\Lambda)$$

for point sets S and lattices Λ in the n-dimensional space F^n with coordinates in a field F of formal Laurent series in one indeterminate. It will, however, be assumed that

$$0 < \Delta(S) < \infty \qquad \text{and} \qquad 0 < m_k(S, \Lambda) < \infty \qquad (k = 1, 2, \ldots, n).$$

The proof is similar to that by Rogers in the real case, but since the distance in F^n arises from a discrete non-archimedean valuation, certain changes are necessary. The quantities $\Delta(S)$, $m_k(S, \Lambda)$, and $d(\Lambda)$ will be defined in the text. Again the constant factor 2^n will be shown to be best possible.

1

Let f be any field, z an indeterminate, $f[z]$ the ring of polynomials and $f(z)$ the field of rational functions in z with coefficients in f.

A special non-archimedean valuation $|x|$ on the field $f(z)$ is given by[1]

$$|0| = 0; \qquad |x| = 2^{N-D} \qquad \text{for} \quad x \neq 0.$$

Here N and D are the degrees of the numerator and of the denominator of the rational function x, respectively. This valuation is discrete, and any collection of its values greater than a positive constant contains a smallest element.

Let F be the completion of the field $f(z)$ relative to the valuation $|x|$. The elements x of F can be written as formal Laurent series

$$x = \sum_{j=-\infty}^{r} a_j z^j$$

with coefficients a_j in f. Here r may be any integer $<0, =0$, or >0. When $x = 0$, all coefficients a_j are zero; while for $x \neq 0$ the integer r will be chosen such that $a_r \neq 0$. The continuation of the valuation $|x|$ from $f(z)$ to F is now defined by

$$|0| = 0; \qquad |x| = 2^r \qquad \text{if} \quad x \neq 0.$$

We define the *integral part* $[x]$ and the *fractional part* (x) of any element x of F by

$$[x] = \sum_{j=0}^{r} a_j z^j \qquad \text{and} \qquad (x) = \sum_{j=-\infty}^{-1} a_j z^j.$$

With this notation for every x in F,

$$x = [x] + (x), \qquad [x] \in f[z], \qquad |(x)| \leqslant \tfrac{1}{2}.$$

2

Denote by n a fixed positive integer and by F^n the n-dimensional space of all points or vectors

$$\mathbf{x} = (x_1, \ldots, x_n), \qquad \mathbf{y} = (y_1, \ldots, y_n), \qquad \mathbf{0} = (0, \ldots, 0),$$

[1] In Mahler [3], I defined the valuation $|x|$ instead by $|x| = e^{N-D}$.

etc., with coordinates (components) in F. The point $\mathbf{0}$ is the *origin of* F^n and plays a specialised role in the theory.

As usual the sum $\mathbf{x}+\mathbf{y}$ and the difference $\mathbf{x}-\mathbf{y}$ of two points $\mathbf{x}$ and $\mathbf{y}$ in F^n are defined by

$$\mathbf{x}+\mathbf{y}=(x_1+y_1,\dots,x_n+y_n) \qquad \text{and} \qquad \mathbf{x}-\mathbf{y}=(x_1-y_1,\dots,x_n-y_n).$$

If further t is any element of F, then $t\mathbf{x}$ denotes the point

$$t\mathbf{x}=(tx_1,\dots,tx_n).$$

A norm $|\mathbf{x}|$ for the points $\mathbf{x}$ in F^n is defined by

$$|\mathbf{x}|=\max(|x_1|,\dots,|x_n|).$$

It has the properties

$$|\mathbf{x}\pm\mathbf{y}|\leqslant\max(|\mathbf{x}|,|\mathbf{y}|), \qquad |t\mathbf{x}|=|t|\,|\mathbf{x}|.$$

The expression

$$\rho(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{y}|$$

represents a distance in F^n and changes this space into a metric space, with all the usual topological implications.

3

The points $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$ in F^n are said to be *linearly dependent* (viz., over F) if there exists elements $t_1,\dots,t_m$ of F not all zero such that

$$t_1\mathbf{x}^{(1)}+\cdots+t_m\mathbf{x}^{(m)}=\mathbf{0},$$

and they are otherwise called *linearly independent*. Further $\mathbf{x}$ is said to be *linearly dependent on* $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$ if elements $s_1,\dots,s_m$ of F exist such that

$$\mathbf{x}=s_1\mathbf{x}^{(1)}+\cdots+s_m\mathbf{x}^{(m)}.$$

If no such $s_1,\dots,s_m$ exist, then $\mathbf{x}$ is called *linearly independent of* $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$.

If the m points $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$ are linearly independent, and if the further point $\mathbf{x}$ is linearly independent of $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$, then the $m+1$ points $\mathbf{x}$, $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$ are linearly independent.

A set of exactly n points

$$\mathbf{x}^{(k)}=(x_{k1},\dots,x_{kn}) \qquad (k=1,2,\dots,n)$$

in F^n is linearly independent if and only if the determinant

$$D(\mathbf{x}^{(1)},\dots,\mathbf{x}^{(n)}) = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

is distinct from zero.

Any $n+1$ points in F^n are linearly dependent.

4

Let S be an arbitrary point set in F^n. If t is any element of F, then tS denotes the set of all points

$$t\mathbf{x}, \qquad \text{where} \quad \mathbf{x} \in S.$$

The largest number of linearly independent points of S is called the *dimension of* S and denoted by $\delta(S)$. We put $\delta(S) = 0$ if S is empty or consists of the single point $\mathbf{0}$. In all other cases $\delta(S)$ is positive and satisfies

$$1 \leqslant \delta(S) \leqslant n.$$

It is obvious that

$$\delta(S) = \delta(tS) \qquad \text{if} \quad t \neq 0 \text{ lies in } F.$$

Of particular importance are two kinds of point sets, the linear manifolds M and the lattices Λ.

A *linear manifold* M is a nonempty point set in F^n with the following two properties:

(a) If $\mathbf{x} \in M$ and $\mathbf{y} \in M$, then also $\mathbf{x}+\mathbf{y} \in M$ and $\mathbf{x}-\mathbf{y} \in M$.

(b) If $\mathbf{x} \in M$ and $t \in F$, then also $t\mathbf{x} \in M$.

The second property implies that all linear manifolds contain the origin $\mathbf{0}$. If M contains other points $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)}$, then also all points

$$t_1\mathbf{x}^{(1)} + \cdots + t_m\mathbf{x}^{(m)}, \qquad \text{where} \quad t_1,\dots,t_m \in F,$$

belong to M, and it is clear that $\delta(M) \geqslant 1$. There exist now $\delta(M) = \delta$ linearly independent points $\mathbf{x}^{(1)},\dots,\mathbf{x}^{(\delta)}$ in M such that M consists exactly of the points

$$t_1\mathbf{x}^{(1)} + \cdots + t_\delta\mathbf{x}^{(\delta)}, \qquad \text{where} \quad t_1,\dots,t_\delta \in F.$$

It follows that

$$tM = M \qquad \textit{for every } t \neq 0 \textit{ in } F.$$

5

A *lattice* Λ in F^n is a point set with the following four properties:

(A) If $\mathbf{x} \in \Lambda$ and $\mathbf{y} \in \Lambda$, then also $\mathbf{x} + \mathbf{y} \in \Lambda$ and $\mathbf{x} - \mathbf{y} \in \Lambda$.

(B) If $x \in \Lambda$ and $u \in f[z]$, then also $u\mathbf{x} \in \Lambda$.

(C) Λ contains n linear independent points; thus $\delta(\Lambda) = n$.

(D) There exists a positive number c such that

$$|\mathbf{x}| \geqslant c \qquad \text{for every point } \mathbf{x} \neq \mathbf{0} \text{ of } \Lambda.$$

It follows from (B) that the origin $\mathbf{0}$ belongs to every lattice.

A set of n linearly independent points $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(n)}$ of Λ is called a *basis of* Λ if every point $\mathbf{x}$ of Λ can be written in the form

$$\mathbf{x} = u_1 \mathbf{x}^{(1)} + \cdots + u_n \mathbf{x}^{(n)}, \qquad \text{where} \quad u_1,\ldots, u_n \in f[z].$$

LEMMA 1. *Every lattice has a basis.*

Proof. Choose for $\mathbf{x}^{(1)}$ a point of Λ distinct from $\mathbf{0}$ such that $|\mathbf{x}^{(1)}|$ is as small as possible. Next, if $2 \leqslant k \leqslant n$ and if the points $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(k-1)}$ of Λ have already been selected, choose for $\mathbf{x}^{(k)}$ a point of Λ which is linearly independent of $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(k-1)}$ and has the property that $|\mathbf{x}^{(k)}|$ is a minimum.

The n lattice points $\mathbf{x}^{(1)},\ldots, \mathbf{x}^{(n)}$ so defined evidently satisfy

$$c \leqslant |\mathbf{x}^{(1)}| \leqslant |\mathbf{x}^{(2)}| \leqslant \cdots \leqslant |\mathbf{x}^{(n)}|.$$

We assert that they form a basis of Λ.

For consider an arbitrary point $\mathbf{x}$ of Λ. This point can certainly be written in the form

$$\mathbf{x} = t_1 \mathbf{x}^{(1)} + \cdots + t_n \mathbf{x}^{(n)}, \qquad \text{where} \quad t_1,\ldots, t_n \in F.$$

Put

$$\mathbf{x}^* = [t_1]\mathbf{x}^{(1)} + \cdots + [t_n]\mathbf{x}^{(n)} \qquad \text{and} \qquad \mathbf{x}^{**} = (t_1)\mathbf{x}^{(1)} + \cdots + (t_n)\mathbf{x}^{(n)}.$$

Then

$$\mathbf{x} = \mathbf{x}^* + \mathbf{x}^{**}$$

and

$$[t_k] \in f[z], \qquad |(t_k)| \leqslant \tfrac{1}{2} \qquad (k = 1, 2, \ldots, n).$$

Therefore, by properties (A) and (B), $\mathbf{x}^*$ and hence also $\mathbf{x}^{**}$ belong to Λ. The assertion will be proved if we can show that $\mathbf{x}^{**} = \mathbf{0}$, or equivalently,

$$(t_1) = \cdots = (t_n) = 0.$$

If this were false, there would exist a suffix k such that

$$(t_k) \neq 0, \qquad (t_{k+1}) = \cdots = (t_n) = 0,$$

so that $\mathbf{x}^{**}$ would be distinct from $\mathbf{0}$ for $k = 1$, but would be linearly independent of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k-1)}$ if $k \geqslant 2$. However,

$$|\mathbf{x}^{**}| = |(t_1)\mathbf{x}^{(1)} + \cdots + (t_k)\mathbf{x}^{(k)}| \leqslant 2^{-1} \max(|\mathbf{x}^{(1)}|, \ldots, |x^{(k)}|) < |\mathbf{x}^{(k)}|,$$

contrary to the definition of $\mathbf{x}^{(k)}$.

6

With every basis $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ of Λ we can associate the determinant $D(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})$; we know that it does not vanish.

LEMMA 2. *If* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *and* $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ *are two bases of* Λ, *then*

$$|D(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})| = |D(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)})|.$$

Proof. There exist two $n \times n$ matrices $(u_{hk})_{h,k=1,2,\ldots,n}$ and $(v_{hk})_{h,k=1,2,\ldots,n}$ with elements u_{hk}, v_{hk} in $f[z]$ such that

$$\mathbf{y}^{(h)} = u_{h1}\mathbf{x}^{(1)} + \cdots + u_{hn}\mathbf{x}^{(n)} \quad \text{and} \quad \mathbf{x}^{(h)} = v_{h1}\mathbf{y}^{(1)} + \cdots + v_{hn}\mathbf{y}^{(n)}$$
$$(h, k = 1, 2, \ldots, n).$$

Denote their determinants by U and V, respectively. Then

$$D(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}) = U \cdot D(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}),$$
$$D(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}) = V \cdot D(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}),$$

from which it follows that

$$UV = 1.$$

Since U and V are polynomials in $f[z]$, U and V necessarily lie in f and are not equal to zero, hence satisfy

$$|U| = |V| = 1.$$

From this the assertion follows immediately.

The lemma implies that if we put

$$d(\Lambda) = |D(\mathbf{x}^{(1)},..., \mathbf{x}^{(n)})|,$$

then $d(\Lambda)$ does not depend on the special basis $\mathbf{x}^{(1)},..., \mathbf{x}^{(n)}$ of Λ. We call $d(\Lambda)$ *the determinant of* Λ. This determinant is always positive and equal to an integral power of 2.

7

From now on pairs (S, Λ) of a point set S and a lattice Λ in F^n will be considered. The lattice is said to be S-*admissible* if none of its points distinct from $\mathbf{0}$ belongs to S. If there are no S-admissible lattices, write

$$\Delta(S) = \infty.$$

Otherwise define $\Delta(S)$ as the lower bound

$$\Delta(S) = \inf d(\Lambda)$$

extended over all S-admissible lattices Λ.

Possibly $\Delta(S)$ is equal to 0; this happens when there are S-admissible lattices Λ with arbitrarily small determinant $d(\Lambda)$.

There remains the case when

$$0 < \Delta(S) < \infty.$$

Now the lower bound $\Delta(S)$ is attained, and there is at least one S-admissible lattice Λ satisfying

$$d(\Lambda) = \Delta(S).$$

Such a lattice is called a *critical lattice of S*.

8

We continue to consider the pair (S, Λ) of a point set S and a lattice Λ in F^n. Let k run over the integers from 1 to n. Denote by Σ_k the set of all integers N for which the set $z^N S$ contains k linearly independent points of Λ. It is obvious that

$$\Sigma_1 \supset \Sigma_2 \supset \cdots \supset \Sigma_n. \tag{1}$$

If Σ_k is empty, write

$$N_k = +\infty \qquad \text{and} \qquad m_k(S, \Lambda) = 2^{N_k} = +\infty.$$

If Σ_k contains a sequence of integers tending to $-\infty$, write

$$N_k = -\infty \qquad \text{and} \qquad m_k(S, \Lambda) = 2^{N_k} = 0.$$

If neither of these two cases holds, then Σ_k contains a smallest integer, N_k say. Now put

$$m_k(S, \Lambda) = 2^{N_k},$$

so that

$$0 < m_k(S, \Lambda) < \infty.$$

In this third case the set $z^{N_k} S$, but not the set $z^{N_k - 1} S$, contains k linearly independent points of Λ.

It is clear from (1) that always

$$-\infty \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_n \leqslant +\infty, \tag{2}$$

and therefore

$$0 \leqslant m_1(S, \Lambda) \leqslant m_2(S, \Lambda) \leqslant \cdots \leqslant m_n(S, \Lambda) \leqslant \infty. \tag{3}$$

The numbers $m_k(S, \Lambda)$ are called *the successive minima of* S in Λ.

9

Assume now that all the integers N_k are finite, hence that

$$\begin{gathered} -\infty < N_1 \leqslant N_2 \leqslant \cdots \leqslant N_n < +\infty, \\ 0 < m_1(S, \Lambda) \leqslant m_2(S, \Lambda) \leqslant \cdots \leqslant m_n(S, \Lambda) < \infty. \end{gathered} \tag{4}$$

For each $k = 1, 2, \ldots, n$ form the point set

$$I_k = (z^{N_k - 1} S \cap \Lambda) \cup \{\mathbf{0}\}$$

which consists of the origin $\mathbf{0}$ and of all the points of $z^{N_k - 1} S$ which lie in Λ.

As was mentioned, the set $z^{N_k-1}S$ cannot contain more than $k-1$ points of Λ which are linearly independent.

Denote by M_k the linear manifold in F^n which is spanned by the points of I_k. Its dimension $\delta(M_k) = \delta_k$ satisfies the inequality

$$0 \leqslant \delta_k \leqslant k-1.$$

Choose an arbitrary $\mathbf{x} \neq \mathbf{0}$ of I_k if such a point exists; this point lies of course also in M_k. Since $N_{k+1} \geqslant N_k$ and since therefore $z^{N_{k+1}-N_k} \in f[z]$,

$$z^{N_{k+1}-N_k}\mathbf{x} \in \Lambda.$$

Further $\mathbf{x} \in z^{N_k-1}S$ and therefore

$$z^{N_{k+1}-N_k}\mathbf{x} \in z^{N_{k+1}-N_k} \cdot z^{N_k-1}S = z^{N_{k+1}-1}S.$$

It follows that

$$z^{N_{k+1}-N_k}\mathbf{x} \in I_{k+1}.$$

This relation implies that

$$z^{N_{k+1}-N_k}M_k \subset M_{k+1}.$$

On the other hand M_k is a linear manifold and therefore

$$z^{N_{k+1}-N_k}M_k = M_k.$$

Therefore $M_k \subset M_{k+1}$ and hence

$$M_1 \subset M_2 \subset \cdots \subset M_n.$$

From this set of relations we can deduce that Λ has a basis $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}$ such that the manifolds M_k is for every k spanned by the points

$$\mathbf{0}, \mathbf{x}^{(1)},\ldots, \mathbf{x}^{(\delta_k)}.$$

10

From this basis $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}$ of Λ we derive now a new lattice Λ^* with the basis

$$\mathbf{X}^{(1)} = z^{-N_1+1}\mathbf{x}^{(1)}, \mathbf{X}^{(2)} = z^{-N_2+1}\mathbf{x}^{(2)},\ldots, X^{(n)} = z^{-N_n+1}x^{(n)}$$

and therefore with the determinant

$$d(\Lambda^*) = 2^{-N_1-N_2-\cdots-N_n+n}\, d(\Lambda). \tag{5}$$

LEMMA 3. *The lattice Λ^* is S-admissible.*

Proof. Let the assertion be false, i.e., let there exist a point $\mathbf{X} \neq \mathbf{0}$ which belongs to both S and Λ^*. In terms of the basis of Λ^*,

$$\mathbf{X} = u_1 \mathbf{X}^{(1)} + \cdots + u_n \mathbf{X}^{(n)},$$

where $u_1, \ldots, u_n$ are polynomials not all zero in $f[z]$. Assume, say, that

$$u_k \neq 0, \qquad u_{k+1} = u_{k+2} = \cdots = u_n = 0.$$

The new point

$$\mathbf{Y} = z^{N_k - 1}\mathbf{X} = z^{N_k - N_1} u_1 \mathbf{x}^{(1)} + z^{N_k - N_2} u_2 \mathbf{x}^{(2)} + \cdots + u_k \mathbf{x}^{(k)}$$

belongs to the original lattice Λ because the coefficients

$$z^{N_k - N_1} u_1, z^{N_k - N_2} u_2, \ldots, u_k$$

lie by $N_1 \leqslant N_2 \leqslant \cdots \leqslant N_n$ in the ring $f[z]$. It is further obvious that $\mathbf{Y}$ belongs to the set $z^{N_k - 1}S$. Hence $\mathbf{Y}$ is a point of I_k and so also a point of M_k. However, by $u_k \neq 0$, the point $\mathbf{Y}$ is linearly independent of the $k-1$ basis points $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k-1)}$, from which by $\delta_k \leqslant k-1$ it follows that $\mathbf{Y}$ is also linearly independent of the points $\mathbf{0}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(\delta_k)}$ which span the linear manifold M_k. Hence a contradiction arises, proving the assertion.

The main result of this paper follows now at once.

THEOREM 1. *If the set S and the lattice Λ satisfy condition* (4), *then the following inequality holds*:

$$\Delta(S) \prod_{k=1}^{n} m_k(S, \Lambda) \leqslant 2^n \, d(\Lambda).$$

Proof. Lemma 3 implies that

$$d(\Lambda^*) \geqslant \Delta(S).$$

On substituting here for $d(\Lambda^*)$ its value (5) and noting that $2^{N_k} = m_k(S, \Lambda)$ for all k, the assertion is obtained immediately.

Theorem 1 shows in particular that if (4) holds, then $\Delta(S) < \infty$. In fact, it suffices to assume that N_1 and therefore also $m_1(S, \Lambda)$ are finite. For then no point of Λ distinct from $\mathbf{0}$ lies in $z^{N_1 - 1}S$ so that Λ is $z^{N_1 - 1}S$-admissible. Hence the lattice $z^{-N_1 + 1}\Lambda$ is S-admissible and therefore $\Delta(S) < \infty$.

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It is obvious that every bounded set S has admissible lattices and hence satisfies $\Delta(S) < \infty$. There is also no difficulty in constructing unbounded sets with the same property.

The successive minima $m_k(S, \Lambda)$ depend on both the set S and the lattice Λ and they may well be equal to ∞. Thus, when S lies in a linear manifold M of dimension $\delta \leqslant n-1$, then clearly for every lattice Λ,

$$m_{\delta+1}(S, \Lambda) = m_{\delta+2}(S, \Lambda) = \cdots = m_n(S, \Lambda) = \infty.$$

However, the following simple result holds.

THEOREM 2. *If S has at least one interior point, then*

$$m_k(S, \Lambda) < \infty \qquad (k = 1, 2, \ldots, n)$$

for all lattices Λ.

Proof. To the interior point of S, $\mathbf{X}$ say, there exists a positive number C such that all point $\mathbf{x}$ in F^n satisfying

$$|\mathbf{x} - \mathbf{X}| \leqslant C$$

belong to S. Hence, for every integer N, all points $\mathbf{x}$ in F^n satisfying

$$|\mathbf{x} - \mathbf{z}^N\mathbf{X}| \leqslant 2^N C$$

lie in the set $z^N S$.

Denote now by $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ a basis of Λ and put

$$c = \max(|\mathbf{x}^{(1)}|, \ldots, |\mathbf{x}^{(n)}|);$$

further choose N so large that

$$2^N C \geqslant c.$$

We assert then that the set $z^N S$ contains n linearly independent points of Λ, so that the assertion holds. For just as in the proof of Lemma 1 we can find first a point $\mathbf{x}(0)$ of Λ satisfying

$$|\mathbf{x}(0) - 2^N\mathbf{X}| \leqslant c/2,$$

and then the n further points $\mathbf{x}(k)$ of Λ defined by

$$\mathbf{x}(k) = \mathbf{x}(0) + \mathbf{x}^{(k)} \qquad (k = 1, 2, \ldots, n)$$

have the property

$$|\mathbf{x}(k) - z^N \mathbf{X}| \leqslant c \qquad (k = 1, 2,..., n).$$

All $n + 1$ lattice points $\mathbf{x}(0), \mathbf{x}(1),..., \mathbf{x}(n)$ lie therefore in $z^N S$. But since

$$\mathbf{x}^{(k)} = \mathbf{x}(k) - \mathbf{x}(0) \qquad (k = 1, 2,..., n),$$

certain n of these lattice points are linearly independent.

I know of no similar simple condition under which one or more of the successive minima $m_k(S, \Lambda)$ are distinct from zero. This problem is for unbounded sets S connected to quite difficult questions in Diophantine approximations.

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Let us finally prove that the factor 2^n in inequality (I) is best possible. The proof is an immediate consequence of

THEOREM 3. *The set S of all points $\mathbf{x} = (x_1,..., x_n)$ satisfying*

$$|\mathbf{x}| = \max(|x_1|,..., |x_n|) \leqslant 1$$

has the property that

$$\Delta(S) = 2^n. \tag{6}$$

Let us for the moment assume that this theorem is true, and choose for Λ the lattice with the basis

$$\mathbf{e}^{(1)} = (1, 0,..., 0), \mathbf{e}^{(2)} = (0, 1,..., 0),..., \mathbf{e}^{(n)} = (0, 0,..., 1)$$

and hence with the determinant $d(\Lambda) = 1$. These n basis points belong to S, but not to $z^{-1}S$; therefore

$$m_k(S, \Lambda) = 1 \qquad (k = 1, 2,..., n),$$

which shows that (I) holds with equality.

Proof of Eq. (6). We apply results from Mahler [3]. In the notation of this paper, S is a *convex body of volume* $V = 1$.

Let Λ be an arbitrary lattice of basis $\mathbf{x}^{(1)},..., \mathbf{x}^{(n)}$, where

$$\mathbf{x}^{(k)} = (x_{k1},..., x_{kn}) \qquad (k = 1, 2,..., n)$$

and determinant

$$d(\Lambda) = |D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})|. \tag{7}$$

The general point $\mathbf{x}$ of Λ has the form

$$\mathbf{x} = u_1 \mathbf{x}^{(1)} + \cdots + u_n \mathbf{x}^{(n)}, \qquad \text{where } u_1, \dots, u_n \in f[z].$$

We put

$$\mathbf{u} = (u_1, \dots, u_n)$$

and consider the linear mapping in F^n from $\mathbf{u}$ to $\mathbf{x}$. In the coordinates of $\mathbf{u}$ the lattice Λ is transformed into the lattice Λ^* of basis $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n)}$. The mapping from Λ to Λ^* has the determinant $D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})$ which occurs in (7). The function $|\mathbf{x}|$ takes in the new variables $u_1, \dots, u_n$ the form

$$|\mathbf{x}| = \max_{k=1,2,\dots,n} |u_1 x_{1k} + \cdots + u_n x_{nk}|, = \Psi(\mathbf{u}) \qquad \text{say.}$$

Here $\Psi(\mathbf{u})$ is a *convex distance function* in the terminology of Mahler [3], and the corresponding *convex body* K of all points $\mathbf{u}$ satisfying

$$\Psi(\mathbf{u}) \leqslant 1$$

has by my paper the *volume*

$$V = |D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})|^{-1} = d(\Lambda)^{-1}.$$

Now, by Section 9 of [3], there exist n points $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(n)}$ of determinant 1 with coordinates in $f[z]$ (i.e., these points form a basis of Λ^*) such that

$$\prod_{k=1}^{n} \Psi(\mathbf{u}^{(k)}) = V^{-1} = d(\Lambda). \tag{8}$$

The coordinates of these points are, say,

$$\mathbf{u}^{(k)} = (u_{k1}, \dots, u_{kn}) \qquad (k = 1, 2, \dots, n).$$

We derive from them the points

$$\mathbf{X}^{(k)} = u_{k1} \mathbf{x}^{(1)} + \cdots + i_{kn} \mathbf{x}^{(n)} \qquad (k = 1, 2, \dots, n)$$

which form a basis of the original lattice Λ.

By (8), this basis now satisfies the equation

$$|\mathbf{X}^{(1)}|\,|\mathbf{X}^{(2)}| \cdots |\mathbf{X}^{(n)}| = d(\Lambda).$$

Here each of the factors $|\mathbf{X}^{(1)}|, |\mathbf{X}^{(2)}|, \dots, |\mathbf{X}^{(n)}|$ is an integral power of 2. If now $d(\Lambda) \leqslant 2^{n-1}$, then at least one factor $|\mathbf{X}^{(k)}|$ does not exceed 1 and hence $\mathbf{X}^{(k)}$ lies in S. But $\mathbf{X}^{(k)}$ is certainly distinct from $\mathbf{0}$; therefore Λ is not S-admissible.

It follows that Λ only then can be S-admissible if $d(\Lambda) \geqslant 2^n$, giving the assertion because the lattice of basis $z\mathbf{e}^{(1)}, \ldots, z\mathbf{e}^{(n)}$ and determinant 2^n is S-admissible. This proves the assertion.

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium