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A New Transfer Principle in the Geometry of Numbers

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In my paper (*Proc. Roy. Soc. Edinburgh Sect.* A **64** (1956), 223–238), I gave a general transfer principle in the geometry of numbers which consisted of inequalities linking the successive minima of a convex body in n dimensions with those of a convex body in N dimensions where in general N is greater than n. This result contained in particular my earlier theorem on compound convex bodies (*Proc. London Math. Soc.* (3) **5** (1955), 358–379). In the present paper I apply essentially the same method to prove a new transfer principle which connects the successive minima of a convex body in m dimensions and those of a convex body in n dimensions with the successive minima of a convex body in mn dimensions.

1. Let $m \geqslant 2$ and $n \geqslant 2$ be integers, let $\mathbf{R}^m$ and $\mathbf{R}^n$ be the real m-dimensional and n-dimensional spaces of all points or vectors

$$\mathbf{x} = (x_1, ..., x_m) \qquad \text{and} \qquad \mathbf{y} = (y_1, ..., y_n),$$

respectively, and let $\mathbf{R}^{mn}$ be the real mn-dimensional space of all points or vectors

$$\mathbf{z} = (z_{11}, z_{12}, ..., z_{mn}),$$

where the coordinates

$$z_{hk}, \qquad (h = 1, 2, ..., m, k = 1, 2, ..., n)$$

are arranged in lexicographical order. We denote by

$$\mathbf{u}_1 = (1, ..., 0), ..., \mathbf{u}_m = (0, ..., 1)$$

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the m points in R^m with just one coordinate 1 and all others 0, by

$$\mathbf{v}_1 = (1,..., 0),..., \mathbf{v}_n = (0,..., 1)$$

the analogous points in $\mathbf{R}^n$, and by

$$\mathbf{w}_{hk}, \qquad (h = 1, 2,..., m, k = 1, 2,..., n)$$

the mn points in $\mathbf{R}^{mn}$ which have a coordinate 1 at the place h, k and 0 at all other places. With the usual notation for sums of vectors and for the product of a vector with a scalar, the points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ may then be written as

$$\mathbf{x} = \sum_{h=1}^{m} x_h \mathbf{u}_h, \qquad \mathbf{y} = \sum_{k=1}^{n} y_k \mathbf{v}_k, \qquad \mathbf{z} = \sum_{h=1}^{m} \sum_{k=1}^{n} z_{hk} \mathbf{w}_{hk}.$$

Finally, denote by $\mathbf{L}^m$, $\mathbf{L}^n$, and $\mathbf{L}^{mn}$ the lattices of all points in $\mathbf{R}^m$, $\mathbf{R}^n$, and $\mathbf{R}^{mn}$, respectively, which have integral coordinates. Then the lattice points $\mathbf{u}_h$ form a basis of $\mathbf{L}^m$, the lattice points $\mathbf{v}_k$ a basis of $\mathbf{L}^n$, and the lattice points $\mathbf{w}_{hk}$ form a basis of $\mathbf{L}^{mn}$. All three lattices have the determinant 1.

2. We introduce now the mapping $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^{mn}$ defined by the equations

$$z_{hk} = x_h \cdot y_k, \qquad (h = 1, 2,..., m, k = 1, 2,..., n).$$

We write $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ and note that here the order of $\mathbf{x}$ and $\mathbf{y}$ may *not* be altered.

When $\mathbf{x}$ runs over the whole space $\mathbf{R}^m$ and $\mathbf{y}$ over the whole space R^n, then $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ describes the algebraic manifold in R^{mn}, M say, which is defined by the algebraic equations

$$z_{hk} z_{ij} = z_{hj} z_{ik}, \qquad h, i = 1, 2,..., m, k, j = 1, 2,..., n).$$

Since $\mathbf{u}_h \times \mathbf{v}_k = \mathbf{w}_{hk}$ for $h = 1, 2,..., m$ and $k = 1, 2,..., n$, the manifold M contains the mn unit points $\mathbf{w}_{hk}$ which together span the space $\mathbf{R}^{mn}$.

In the equation $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ the coordinates of $\mathbf{z}$ are bilinear forms in the coordinates of $\mathbf{x}$ and of $\mathbf{y}$ and hence are continuous functions in these coordinates.

3. Denote by

$$\mathbf{A} = (a_{hi}) \qquad \text{and} \qquad \mathbf{B} = (b_{kj})$$

a real non-singular $m \times m$ matrix of determinant

$$a = \det(a_{hi}) \neq 0$$

and a real non-singular $n \times n$ matrix of determinant

$$b = \det(b_{kj}) \neq 0.$$

We associate with $\mathbf{A}$ the non-singular linear transformation of $\mathbf{R}^m$ defined by

$$\mathbf{X} = \mathbf{A}\mathbf{x} = (X_1, \dots, X_m), \qquad \text{where} \quad X_h = \sum_{i=1}^{m} a_{hi} x_i \ \ (h = 1, 2, \dots, m)$$

and with $\mathbf{B}$ the non-singular linear transformation of $\mathbf{R}^n$ defined by

$$\mathbf{Y} = \mathbf{B}\mathbf{y} = (Y_1, \dots, Y_n), \qquad \text{where} \quad Y_k = \sum_{j=1}^{n} b_{kj} y_j \ \ (k = 1, 2, \dots, n).$$

If simultaneously $\mathbf{A}$ is applied to $\mathbf{x}$ and $\mathbf{B}$ to $\mathbf{y}$, then $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ is changed into

$$\mathbf{Z} = \mathbf{A}\mathbf{x} \times \mathbf{B}\mathbf{y} = \mathbf{X} \times \mathbf{Y} = (Z_{11}, Z_{12}, \dots, Z_{mn}),$$

where the new coordinates Z_{hk} are again numbered lexicographically and have the values

$$Z_{hk} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{hi} b_{kj} z_{ij}, \qquad (h = 1, 2, \dots, m, \ k = 1, 2, \dots, n).$$

This is again a linear transformation of $\mathbf{R}^{mn}$ defined by

$$\mathbf{Z} = \mathbf{C}\mathbf{z}, \qquad \text{where} \quad \mathbf{C} = (c_{hi,kj})$$

$$\text{and} \qquad c_{hi,kj} = a_{hi} b_{kj}, \quad (h, i = 1, 2, \dots, m, \ k, j = 1, 2, \dots, n).$$

As is well known, the $mn \times mn$ matrix $\mathbf{C}$ has the determinant

$$c = \det(c_{hi,kj}) = a^n b^m \neq 0,$$

so that also $\mathbf{C}$ is non-singular. We shall use the notation $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

4. A "body" is a point set with interior points and a "convex body" a closed bounded convex body which is symmetric in the coordinate origin $\mathbf{o} = (0, \dots, 0)$, and for which $\mathbf{o}$ is an interior point.

Let K^m be any convex body in $\mathbf{R}^m$ and K^n any convex body in $\mathbf{R}^n$. As the point $\mathbf{x}$ runs over the whole of K^m and the point $\mathbf{y}$ over the whole of K^n, the product point

$$\mathbf{z} = \mathbf{x} \times \mathbf{y} \tag{1}$$

describes a certain point set, Σ say, which is a subset of the manifold M. Denote by K^{mn} the convex hull of Σ so that K^{mn} is a convex point set in $\mathbf{R}^{mn}$. We shall use the notation

$$K^{mn} = K^m \times K^n.$$

LEMMA 1. *The point set K^{mn} is a convex body.*

Proof. Since the mapping (1) is continuous, both Σ and K^{mn} are bounded closed point sets; further K^{mn}, as already said, is convex.

Next, if $\mathbf{x}$ is any point of K^m, then also $-\mathbf{x}$ belongs to K^m. Now

$$(-\mathbf{x}) \times \mathbf{y} = -\mathbf{x} \times \mathbf{y}.$$

It follows that if $\mathbf{z}$ is any point of K^{mn}, then also $-\mathbf{z}$ belongs to K^{mn}, and hence K^{mn} is symmetric in $\mathbf{o}$.

Finally, $\mathbf{o}$ is an interior point of K^{mn}. For both K^m and K^n contain the origins of $\mathbf{R}^m$ and of $\mathbf{R}^n$, respectively, as interior points. This implies that there exist two positive constants δ and ε such that K^m contains the $2m$ points

$$\pm \delta \cdot \mathbf{u}_h \qquad (h = 1, 2, \ldots, m),$$

K^n contains the $2n$ points

$$\pm \varepsilon \cdot \mathbf{v}_k \qquad (k = 1, 2, \ldots, n),$$

and therefore both the set Σ and the convex body K^{mn} contain the $2mn$ points

$$\pm \delta\varepsilon \cdot \mathbf{w}_{hk} \qquad (h = 1, 2, \ldots, m, \; k = 1, 2, \ldots, n).$$

But then, by convexity, K^{mn} contains all points of the form

$$\delta\varepsilon \sum_{h=1}^{m} \sum_{k=1}^{n} t_{hk} \mathbf{w}_{hk},$$

where $t_{11}, t_{12}, \ldots, t_{mn}$ denote any real numbers satisfying the inequality

$$\sum_{h=1}^{m} \sum_{k=1}^{n} |t_{hk}| \leqslant 1.$$

Since the mn points $\mathbf{w}_{hk}$ span the space R^{mn}, it follows that $\mathbf{o}$ is an interior point of K^{mn}. This concludes the proof.

5. Let again $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be the transformations in Section 3, and let further $K^{mn} = K^m \times K^n$. Put

$$K^{om} = \mathbf{A}K^m, \qquad K^{on} = \mathbf{B}K^n, \qquad \text{and} \qquad K^{omn} = \mathbf{C}K^{mn}.$$

Here $\mathbf{A}K^m$ is to consist of all points $\mathbf{A}\mathbf{x}$, where $\mathbf{x}$ belongs to K^m, and similarly for $\mathbf{B}K^n$ and $\mathbf{C}K^{mn}$. Since we are dealing with affine transformations, K^{om}, K^{on}, and K^{omn} are again convex bodies, and moreover

$$K^{omn} = K^{om} \times K^{on}.$$

Next denote by

$$J^{(m)} = \int_{K^m} \cdots \int dx_1 \cdots dx_m, \qquad J^{(n)} = \int_{K^n} \cdots \int dy_1 \cdots dy_n,$$

$$J^{(m,n)} = \int_{K^{mn}} \cdots \int dz_{11}\, dz_{12} \cdots dz_{mn}$$

the volumes of K^m, K^n, and K^{mn} in their respective spaces and by $J^{o(m)}$, $J^{o(n)}$, and $J^{o(m,n)}$ the analogous volumes of K^{om}, K^{on}, and K^{omn}, respectively. Then evidently

$$J^{o(m)} = aJ^{(m)}, \qquad J^{o(n)} = bJ^{(n)}, \qquad \text{and} \qquad J^{o(m,n)} = cJ^{(m,n)} = a^n b^m J^{(m,n)}.$$

Therefore

$$J^{o(m)n} J^{o(n)m} / J^{o(m,n)} = J^{(m)n} J^{(n)m} / J^{(m,n)}$$

so that this quotient of volumes is invariant under the transformations.

6. Consider first a special case. Denote by G^m and G^n the unit ball $|\mathbf{x}| \leqslant 1$ in $\mathbf{R}^m$ and the unit ball $|\mathbf{y}| \leqslant 1$ in $\mathbf{R}^n$ and define a convex body G^{mn} by the equation

$$G^{mn} = G^m \times G^n.$$

This body G^{mn} is rather complicated and is in fact the convex hull of the intersection of the unit ball $|z| \leqslant 1$ in $\mathbf{R}^{mn}$ with the manifold M. Let $g^{(m)}$, $g^{(n)}$, and $g^{(m,n)}$ be the volumes of G^m, G^n, and G^{mn}, respectively. These three volumes depend only on the degrees m and n.

Next let E^m be any ellipsoid in $\mathbf{R}^m$ and E^n any ellipsoid in R^n, both with their centres at the origins of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and let E^{mn} be the convex body in $\mathbf{R}^{mn}$ defined by

$$E^{mn} = E^m \times E^n.$$

The volumes of E^m, E^n, and E^{mn} will be denoted by $e^{(m)}$, $e^{(n)}$, and $e^{(m,n)}$, respectively.

LEMMA 2. *There exists a positive number c_1 depending only on m and n such that*

$$e^{(m,n)} = c_1 e^{(m)n} \cdot e^{(n)m}.$$

Proof. There exist two non-singular linear transformations $\mathbf{A}$ in $\mathbf{R}^m$ and $\mathbf{B}$ in $\mathbf{R}^n$ such that

$$E^m = \mathbf{A}G^m \qquad \text{and} \qquad E^n = \mathbf{B}G^n$$

and therefore

$$E^{mn} = \mathbf{C}G^{mn},$$

where $\mathbf{C}$ is derived from $\mathbf{A}$ and $\mathbf{B}$ as in Section 3. It follows now from Section 5 that

$$e^{(m)} = ag^{(m)}, \qquad e^{(n)} = bg^{(n)}, \qquad e^{(m,n)} = cg^{(m,n)} = a^n b^m g^{(m,n)},$$

whence the assertion on putting

$$c_1 = g^{(m,n)}/g^{(m)n} g^{(n)m}.$$

7. If S is any point set and $s > 0$ is a scalar, denote as usual by sS the set of all points sP where P runs over S. It is obvious that in this notation, for every convex body K^m in $\mathbf{R}^m$ and every convex body K^n in $\mathbf{R}^n$ and for any two positive numbers s and t, from the definition of $K^m \times K^n$,

$$sK^m \times tK^n = stK^{mn}.$$

By the same definition, if K_1^m and K_2^m are two convex bodies in $\mathbf{R}^m$, and K_1^n and K_2^n are two convex bodies in $\mathbf{R}_n$, such that

$$K_1^m \subset K_2^m \qquad \text{and} \qquad K_1^n \subset K_2^n$$

and if further

$$K_1^{mn} = K_1^m \times K_1^n \qquad \text{and} \qquad K_2^{mn} = K_2^m \times K_2^n,$$

then also

$$K_1^{mn} \subset K_2^{mn}.$$

Let now again K^m, K^n, and $K^{mn} = K^m \times K^n$ be the original convex bodies in $\mathbf{R}^m$, $\mathbf{R}^n$, and $\mathbf{R}^{mn}$, respectively, and let $J^{(m)}$, $J^{(n)}$, and $J^{(m,n)}$ be their volumes. Then the following result holds:

THEOREM 1. *There exist two positive constants c_2 and c_3 which depend only on the dimensions m and n such that*

$$c_2 J^{(m)n} J^{(n)m} \leqslant J^{(m,n)} \leqslant c_3 J^{(m)n} J^{(n)m}.$$

Proof. By a theorem by John [1] there exists in $\mathbf{R}^m$ an ellipsoid E^m and in $\mathbf{R}^n$ an ellipsoid E^n such that

$$m^{-1/2} E^m \subset K^m \subset E^m \qquad \text{and} \qquad n^{-1/2} E^n \subset K^n \subset E^n,$$

hence that

$$(mn)^{-1/2} E^{mn} \subset K^{mn} \subset E^{mn}.$$

Let again $J^{(m)}$, $J^{(n)}$, $J^{(m,n)}$, $e^{(m)}$, $e^{(n)}$, $e^{(m,n)}$ be the volume of K^m, K^n, K^{mn}, E^m, E^n, and E^{mn}, respectively. Then $m^{-1/2}E^m$ has the volume $m^{-m/2}e^{(m)}$, $n^{-1/2}E^n$ has the volume $n^{-n/2}e^{(n)}$, and $(mn)^{-1/2}E^{mn}$ has the volume $(mn)^{-mn/2}e^{(m,n)}$. By what has already been proved,

$$m^{-m/2} e^{(m)} \leqslant J^{(m)} \leqslant e^{(m)}, \qquad n^{-n/2} e^{(n)} \leqslant J^{(n)} \leqslant e^{(n)},$$
$$(mn)^{-mn/2} e^{(m,n)} \leqslant J^{(m,n)} \leqslant e^{(m,n)}.$$

Therefore by Lemma 2,

$$J^{(m,n)}/J^{(m)n} J^{(n)m} \leqslant e^{(m,n)} (m^{-m/2} e^{(m)})^{-n} \, (n^{-n/2} e^{(n)})^{-m} \leqslant c_1 (mn)^{mn}$$

and

$$J^{(m,n)}/J^{(m)n} J^{(n)m} \geqslant (mn)^{-mn/2} e^{(m,n)}/e^{(m)n} e^{(n)m} = c_1 (mn)^{-mn/2}.$$

On putting $c_2 = c_1 (mn)^{-mn/2}$ and $c_3 = c_1 (mn)^{mn}$, this proves the assertion.

8. To each of the three convex bodies K^m, K^n, and K^{mn} corresponds a convex distance function, $F^{(m)}(\mathbf{x})$ in $\mathbf{R}^m$, $F^{(n)}(\mathbf{y})$ in $\mathbf{R}^n$, and $F^{(m,n)}(\mathbf{z})$ in $\mathbf{R}^{mn}$, respectively. Here, e.g., $F^{(m)}(\mathbf{x})$ is defined by

$$0 \leqslant F^{(m)}(\mathbf{x}) \leqslant 1 \qquad \text{if and only if} \quad \mathbf{x} \in K^m,$$

or more explicitly,

$$\mathbf{x} \in sK^m \quad \text{if} \quad |s| \geqslant F^{(m)}(\mathbf{x}) \quad \text{and} \quad \mathbf{x} \notin sK^m \quad \text{if} \quad |s| < F^{(m)}(\mathbf{x}).$$

Further,

$$F^{(m)}(\mathbf{o}) = 0, \qquad F^{(m)}(\mathbf{x}) > 0 \qquad \text{if} \quad \mathbf{x} \neq \mathbf{o};$$

$$F^{(m)}(s\mathbf{x}) = |s|\, F(\mathbf{x}) \qquad \text{for all real } s \text{ and } \mathbf{x} \in \mathbf{R}^m;$$

$$F^{(m)}(\mathbf{x}_1 + \mathbf{x}_2) \leqslant F^{(m)}(\mathbf{x}_1) + F^{(m)}(\mathbf{x}_2).$$

Analogous properties are satisfied by the two other distance functions $F^{(n)}(\mathbf{y})$ and $F^{(m,n)}(\mathbf{z})$, in particular,

$$0 \leqslant F^{(n)}(\mathbf{y}) \leqslant 1 \qquad \text{if and only if} \quad \mathbf{y} \in K^{(n)},$$

$$0 \leqslant F^{(m,n)}(\mathbf{z}) \leqslant 1 \qquad \text{if and only if} \quad \mathbf{z} \in K^{mn}.$$

LEMMA 3. *If* $\mathbf{x} \in \mathbf{R}^m$ *and* $\mathbf{y} \in \mathbf{R}^n$ *and therefore* $\mathbf{z} = \mathbf{x} \times \mathbf{y} \in \mathbf{R}^{mn}$, *then*

$$F^{(m,n)}(\mathbf{z}) \leqslant F^{(m)}(\mathbf{x})\, F^{(n)}(\mathbf{y}).$$

Proof. The assertion is obvious if $\mathbf{x} = \mathbf{o}$ or $\mathbf{y} = \mathbf{o}$ and therefore $\mathbf{z} = 0$. Let therefore $\mathbf{x} \neq \mathbf{o}$ and $\mathbf{y} \neq \mathbf{o}$ so that

$$F^{(m)}(\mathbf{x}) > 0 \qquad \text{and} \qquad F^{(n)}(\mathbf{y}) > 0.$$

On putting

$$\mathbf{x}^0 = F^{(m)}(\mathbf{x})^{-1}\, \mathbf{x} \qquad \text{and} \qquad \mathbf{y}^0 = F^{(n)}(\mathbf{y})^{-1}\, \mathbf{y},$$

evidently $F^{(m)}(\mathbf{x}^0) = 1$ and $F^{(n)}(\mathbf{y}^0) = 1$ and therefore $\mathbf{x}^0 \in K^m$ and $\mathbf{y}^0 \in K^n$. On defining $\mathbf{z}^0$ now by $\mathbf{z}^0 = \mathbf{x}^0 \times \mathbf{y}^0$,

$$\mathbf{z}^0 = \mathbf{x}^0 \times \mathbf{y}^0 = F^{(m)}(\mathbf{x})^{-1}\, F^{(n)}(\mathbf{y})^{-1}\, \mathbf{x} \times \mathbf{y} = F^{(m)}(\mathbf{x})^{-1}\, F^{(n)}(\mathbf{y})^{-1}\, \mathbf{z}.$$

Since $\mathbf{x}^0 \in K^m$ and $\mathbf{y}^0 \in K^n$, also $\mathbf{z}^0 \in K^{mn}$ and therefore $F^{(m,n)}(\mathbf{z}^0) \leqslant 1$. But

$$F^{(m,n)}(\mathbf{z}^0) = F^{(m)}(\mathbf{x})^{-1}\, F^{(n)}(\mathbf{y})^{-1}\, F^{(m,n)}(\mathbf{z}),$$

whence the assertion.

9. We combine the results so far obtained with Minkowski's theorem on the successive minima of a convex body in a lattice (Minkowski [4]).

This theorem will be applied three times, to K^m relative to the lattice L^m

in $\mathbf{R}^m$, to K^n relative to the lattice L^n in $\mathbf{R}^n$, and to K^{mn} relative to the lattice L^{mn} in $\mathbf{R}^{mn}$. By this theorem, there exist then

m linearly independent points $\mathbf{x}^1,\ldots,\mathbf{x}^m$ in L^m,

n linearly independent points $\mathbf{y}^1,\ldots,\mathbf{y}^n$ in L^n,

mn linearly independent points $\mathbf{z}^1,\ldots,\mathbf{z}^{mn}$ in L^{mn},

with the corresponding successive minima

$$\mu_h^{(m)} = F^{(m)}(\mathbf{x}^h), \qquad (h = 1, 2,\ldots, m),$$
$$\mu_k^{(n)} = F^{(n)}(\mathbf{y}^k), \qquad (k = 1, 2,\ldots, n),$$
$$\mu_l^{(m,n)} = F^{(m,n)}(\mathbf{z}^l), \qquad (l = 1, 2,\ldots, mn),$$

such that the following properties hold:

(i)

$$0 < \mu_1^{(m)} \leqslant \mu_2^{(m)} \leqslant \cdots \leqslant \mu_m^{(m)}, \qquad \frac{2^m}{m!} \leqslant J^{(m)} \mu_1^{(m)} \cdots \mu_m^{(m)} \leqslant 2^m,$$

$$0 < \mu_1^{(n)} \leqslant \mu_2^{(n)} \leqslant \cdots \leqslant \mu_n^{(n)}, \qquad \frac{2^n}{n!} \leqslant J^{(n)} \mu_1^{(n)} \cdots \mu_n^{(n)} \leqslant 2^n,$$

$$0 < \mu_1^{(m,n)} \leqslant \mu_2^{(m,n)} \leqslant \cdots \leqslant \mu_{mn}^{(m,n)}, \qquad \frac{2^{mn}}{(mn)!} \leqslant J^{(m,n)} \mu_1^{(m,n)} \cdots \mu_{mn}^{(m,n)} \leqslant 2^{mn}.$$

(ii) If $\mathbf{X}^1,\ldots,\mathbf{X}^m$ are m linearly independent points in L^m, $\mathbf{Y}^1,\ldots,\mathbf{Y}^n$ n linearly independent points in L^n, and $\mathbf{Z}^1,\ldots,\mathbf{Z}^{mn}$ mn linearly independent points in L^{mn}, and if these points are ordered such that

$$F^{(m)}(\mathbf{X}^1) \leqslant F^{(m)}(\mathbf{X}^2) \leqslant \cdots \leqslant F^{(m)}(\mathbf{X}^m),$$
$$F^{(n)}(\mathbf{Y}^1) \leqslant F^{(n)}(\mathbf{Y}^2) \leqslant \cdots \leqslant F^{(n)}(\mathbf{Y}^n),$$
$$F^{(m,n)}(\mathbf{Z}^1) \leqslant F^{(m,n)}(\mathbf{Z}^2) \leqslant \cdots \leqslant F^{(m,n)}(\mathbf{Z}^{mn}),$$

then

$$F^{(m)}(\mathbf{X}^h) \geqslant \mu_h^{(m)}, \qquad (h = 1, 2,\ldots, m),$$
$$F^{(n)}(\mathbf{Y}^k) \geqslant \mu_k^{(n)}, \qquad (k = 1, 2,\ldots, n),$$
$$F^{(m,n)}(\mathbf{Z}^l) \geqslant \mu_l^{(m,n)}, \qquad (l = 1, 2,\ldots, mn).$$

Here, in the inequalities (i), the factors $J^{(m)}$, $J^{(n)}$, and $J^{(m,n)}$ are again the volumes of the convex bodies K^m, K^n, and K^{mn}, respectively. We deduce from these inequalities that the quotient

$$q = \mu_1^{(m,n)} \cdots \mu_{mn}^{(m,n)} (\mu_1^{(m)} \cdots \mu_m^{(m)})^{-n} (\mu_1^{(n)} \cdots \mu_n^{(n)})^{-m}$$

satisfies the inequalities

$$\frac{2^{mn}}{(mn)!} (2^m)^{-n} (2^n)^{-m} \leqslant J^{(m,n)} J^{(m)-n} J^{(n)-m} \cdot q \leqslant 2^{mn} \left(\frac{2^m}{m!}\right)^{-n} \left(\frac{2^n}{n!}\right)^{-m}.$$

Here apply Theorem 1 to the quotient $J^{(m,n)} J^{(m)-n} J^{(n)-m}$ and put

$$c_4 = (c_3 (mn)!\, 2^{mn})^{-1} \qquad \text{and} \qquad c_5 = (m!)^n (n!)^n (c_3 2^{mn})^{-1}.$$

We obtain then the following result:

LEMMA 4. *There exist two positive constants c_4 and c_5 which depend only on m and n such that*

$$c_4 (\mu_1^{(m)} \cdots \mu_m^{(m)})^n (\mu_1^{(n)} \cdots \mu_n^{(n)})^m \leqslant \mu_1^{(m,n)} \cdots \mu_{mn}^{(m,n)}$$
$$\leqslant c_5 (\mu_1^{(m)} \cdots \mu_m^{(m)})^n (\mu_1^{(n)} \cdots \mu_n^{(n)})^m.$$

10. Let again $\mathbf{x}^h$ $(h = 1, 2, \ldots, m)$ be m linearly independent points in L^m and $\mathbf{y}^k$ $(k = 1, 2, \ldots, n)$, n linearly independent points in L^n at which the successive minima $\mu_h^{(m)}$ and $\mu_k^{(n)}$ are attained. Then the mn product points

$$\mathbf{Z}^{hk} = \mathbf{x}^h \times \mathbf{y}^k, \qquad (h = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n)$$

lie in the lattice L^{mn} and, moreover, they are linearly independent. For there are two non-singular transformations $\mathbf{A}$ and $\mathbf{B}$ as in Section 2 such that

$$\mathbf{x}^h = \mathbf{A}\mathbf{u}_h \ (h = 1, 2, \ldots, m) \qquad \text{and} \qquad \mathbf{y}^k = \mathbf{B}\mathbf{v}_k \ (k = 1, 2, \ldots, n).$$

Further, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is non-singular, and

$$\mathbf{Z}^{hk} = \mathbf{C}\mathbf{w}_{hk}, \qquad (h = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n),$$

where the mn unit points $\mathbf{w}_{hk}$ span the space R^{mn}.

Put

$$f_{hk}^{(m,n)} = F^{(m,n)}(\mathbf{Z}^{hk}), \qquad (h = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n)$$

and denote for $l = 1, 2, \ldots, mn$ by $f_l^{(m,n)}$ the same quantities $f_{hk}^{(m,n)}$ ordered according to size,

$$f_1^{(m,n)} \leqslant f_2^{(m,n)} \leqslant \cdots \leqslant f_{mn}^{(m,n)}. \tag{2}$$

This ordering (which will not be unique if several of the values $f_{hk}^{(m,n)}$ are equal) establishes thus a 1-to-1 correspondence

$$l \leftrightarrow (h, k)$$

between the integers l in $1 \leqslant l \leqslant mn$ and the pairs of integers (h, k) with $1 \leqslant h \leqslant m$, $1 \leqslant k \leqslant n$.

From property (ii) of the successive minima $\mu_l^{(m,n)}$ and from the ordering (2) it follows that

$$f_l^{(m,n)} \geqslant \mu_l^{(m,n)}, \qquad (l = 1, 2, \ldots, mn).$$

On the other hand, by Lemma 3,

$$f_l^{(m,n)} = F^{(m,n)}(\mathbf{Z}^{hk}) \leqslant F^{(m)}(\mathbf{x}^h)\, F^{(n)}(\mathbf{y}^k) = \mu_h^{(m)} \mu_k^{(n)}.$$

We obtain therefore the system of mn inequalities

(iii) $\mu_l^{(m,n)} \leqslant \mu_h^{(m)} \mu_k^{(n)}$ for $l \leftrightarrow (h, k)$,

from which, on multiplying over all suffixes l, it follows in particular that

$$\mu_1^{(m,n)} \cdots \mu_{mn}^{(m,n)} \leqslant (\mu_1^{(m)} \cdots \mu_m^{(m)})^n\, (\mu_1^{(n)} \cdots \mu_n^{(n)})^m,$$

which is slightly better than the right-hand inequality given by Lemma 4. A valid inequality is also obtained if on the left-hand side of this formula the factor $\mu_l^{(m,n)}$ is omitted while the right-hand side is divided by the corresponding product $\mu_h^{(m)} \mu_k^{(n)}$, where again $l \leftrightarrow (h, k)$. On dividing now the left-hand formula in Lemma 4 by this new inequality, it follows that

(iv) $\mu_l^{(m,n)} \geqslant c_4 \mu_h^{(m)} \mu_k^{(n)}$, for $l \leftrightarrow (h, k)$.

We have so obtained the following result:

THEOREM 2. *There exists a constant $c_4 > 0$ depending only on m and n, with the following property: Denote by $\mu_h^{(m)}$, $(h = 1, 2, \ldots, m)$, the successive minima of the convex body K^m in $\mathbf{R}^m$, by $\mu_k^{(n)}$ $(k = 1, 2, \ldots, n)$, the successive minima of the convex body K^n in $\mathbf{R}^n$, and by $\mu_l^{(m,n)}$ $(l = 1, 2, \ldots, mn)$, the successive minima of the convex body $K^{mn} = K^m \times K^n$ in $\mathbf{R}^{mn}$. Let further $p_l^{(m,n)}$ $(l = 1, 2, \ldots, mn)$, be the mn products*

$$\mu_h^{(m)} \mu_k^{(n)}, \qquad (h = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n)$$

numbered in order of increasing size,

$$p_1^{(m,n)} \leqslant p_2^{(m,n)} \leqslant \cdots \leqslant p_{mn}^{(m,n)}.$$

Then

$$c_4\, p_l^{(m,n)} \leqslant \mu_l^{(m,n)} \leqslant p_l^{(m,n)} \qquad (l = 1, 2, \ldots, mn).$$

Hence in particular,

$$c_4\mu_1^{(m)}\mu_1^{(n)} \leqslant \mu_1^{(m,n)} \leqslant \mu_1^{(m)}\mu_1^{(n)}, \qquad c_4\mu_m^{(m)}\mu_n^{(n)} \leqslant \mu_{mn}^{(m,n)} \leqslant \mu_m^{(m)}\mu_n^{(n)}.$$

11. By means of Theorem 2 we shall finally prove a property of the successive minima of convex bodies defined by linear inequalities. The two special convex distance functions

$$F_0^{(m)}(\mathbf{x}) = \max(|x_1|, \ldots, |x_m|) \qquad \text{and} \qquad F_0^{(n)}(\mathbf{y}) = \max(|y_1|, \ldots, |y_n|)$$

generate the convex bodies

$$K_0^m: F_0^{(m)}(\mathbf{x}) \leqslant 1 \quad \text{in } \mathbf{R}^m \qquad \text{and} \qquad K_0^n: F_0^{(n)}(\mathbf{y}) \leqslant 1 \quad \text{in } \mathbf{R}^n,$$

which are generalized cubes of side 2 with their centres at the origin of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. The product body

$$K_0^{*mn} = K_0^m \times K_0^n$$

in $\mathbf{R}^{mn}$ is rather complicated. If $F_0^{*(m,n)}(\mathbf{z})$ is its distance function, then K_0^{*mn} consists of the points $\mathbf{z} \in \mathbf{R}^{mn}$ for which

$$F_0^{*(m,n)}(\mathbf{z}) \leqslant 1.$$

We introduce the further distance function

$$F_0^{(m,n)}(\mathbf{z}) = \max(|z_{11}|, |z_{12}|, \ldots, |z_{mn}|)$$

and the corresponding convex body

$$K_0^{mn}: F_0^{(m,n)}(\mathbf{z}) \leqslant 1 \qquad \text{in } \mathbf{R}^{mn},$$

which is again a generalised cube of side 2 with centre at the origin. It is easily seen that

$$K_0^{*mn} \subset K_0^{mn}$$

and therefore

$$F_0^{(m,n)}(\mathbf{z}) \leqslant F_0^{*(m,n)}(z) \qquad \text{for all} \quad \mathbf{z} \in \mathbf{R}^{mn}. \tag{I}$$

Further, the origin $\mathbf{o}$ is an interior point of K_0^{*mn}. This implies that there is a constant $c_6 > 0$ depending only on m and n such that all points $\mathbf{z}$ satisfying

$$F_0^{(m,n)}(\mathbf{z}) \leqslant 1/c_6$$

belong to K_0^{*mn}, hence that

$$K_0^{mn} \subset c_6 K_0^{*mn},$$

and therefore

$$F_0^{*(m,n)}(\mathbf{z}) \leqslant c_6 F_0^{(m,n)}(\mathbf{z}) \qquad \text{for all} \quad \mathbf{z} \subset \mathbf{R}^{mn}. \tag{II}$$

12. Denote again by

$$\mathbf{A} = (a_{hi}) \qquad \text{and} \qquad \mathbf{B} = (b_{kj})$$

a real $m \times m$ matrix and a real $n \times n$ matrix, and by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (c_{hi,kj}), \qquad \text{where} \quad c_{hi,kj} = a_{hi} b_{kj},$$

the $mn \times mn$ matrix formed from $\mathbf{A}$ and $\mathbf{B}$. It suffices to consider the case when all three matrices have the determinants 1,

$$a = 1, \qquad b = 1, \qquad c = a^n b^m = 1.$$

The four new distance functions

$$F^{(m)}(\mathbf{x}) = F_0^{(m)}(\mathbf{A}\mathbf{x}) \qquad \text{in } \mathbf{R}^m, \qquad F^{(n)}(\mathbf{y}) = F_0^{(n)}(\mathbf{B}\mathbf{y}) \qquad \text{in } \mathbf{R}^n,$$

and

$$F^{*(m,n)}(\mathbf{z}) = F_0^{*(m,n)}(\mathbf{C}\mathbf{z}) \qquad \text{and} \qquad F^{(m,n)}(\mathbf{z}) = F_0^{(m,n)}(\mathbf{C}\mathbf{z}) \qquad \text{in } \mathbf{R}^{mn}$$

define the convex bodies

$$K^m\colon F^{(m)}(\mathbf{x}) \leqslant 1 \qquad \text{in } \mathbf{R}^m, \qquad K^n\colon F^{(n)}(\mathbf{y}) \leqslant 1 \qquad \text{in } \mathbf{R}^n,$$

and

$$K^{*mn}\colon F^{*(m,n)}(\mathbf{z}) \leqslant 1 \qquad \text{and} \qquad K^{mn}\colon F^{(m,n)}(\mathbf{z}) \leqslant 1 \qquad \text{in } \mathbf{R}^{mn}.$$

Of these bodies K^m, K^n, and K^{mn} are generalised parallelepipeds with their centres at the origin, but the body

$$K^{*mn} = K^m \times K^n$$

is more complicated.

In any case, inequalities (I) and (II) of the last section imply that

$$F^{(m,n)}(\mathbf{z}) \leqslant F^{*(m,n)}(\mathbf{z}) \leqslant c_6 F^{(m,n)}(\mathbf{z}) \qquad \text{for all points } \mathbf{z} \in \mathbf{R}^{mn}. \tag{III}$$

With a slight change of notation, let

$$\mu_h^{(m)}, \qquad \mu_k^{(n)}, \qquad \mu_l^{*(m,n)}, \qquad \text{and} \qquad \mu_l^{(m,n)}$$

be the successive minima of K^m, K^n, K^{*mn}, and K^{mn} in the lattices L^m, L^n, and L^{mn}, respectively. Further denote by

$$\mathbf{z}^{*l} \quad \text{and} \quad \mathbf{z}^l \quad (l=1, 2, ..., mn)$$

two systems of mn linearly independent lattice points in L^{mn} such that

$$\mu_l^{*(m,n)} = F^{*(m,n)}(\mathbf{z}^{*l}) \quad \text{and} \quad \mu_l^{(m,n)} = F^{(m,n)}(\mathbf{z}^l) \quad (l=1, 2, ..., mn).$$

Here, by Theorem 2, if $p_l^{(m,n)}$ has the same meaning as before,

$$c_4\, p_l^{(m,n)} \leqslant \mu_l^{*(m,n)} \leqslant p_l^{(m,n)} \quad (l=1, 2, ..., mn).$$

Further, by property (ii) of the successive minima,

$$F^{*(m,n)}(\mathbf{z}^l) \geqslant F^{*(m,n)}(\mathbf{z}^{*l}), \quad F^{(m,n)}(\mathbf{z}^{*l}) \geqslant F^{(m,n)}(\mathbf{z}^l) \quad (l=1, 2, ..., mn),$$

and therefore by (III)

$$(c_4/c_6)\, p_l^{(m,n)} \leqslant (1/c_6)\, \mu_l^{*(m,n)} \leqslant \mu_l^{(m,n)} \leqslant \mu_l^{*(m,n)} \leqslant p_l^{(m,n)} \quad (l=1, 2, ..., mn).$$

We thus arrive at the following result:

THEOREM 3. *There exists a constant $c_7 > 0$ depending only on m and n, with the following property: Denote by $\mathbf{A} = (a_{hi})$ a real $m \times m$ matrix and by $\mathbf{B} = (b_{kj})$ a real $n \times n$ matrix, and let $\mathbf{C}$ be the $mn \times mn$ matrix*

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (c_{hi,kj}), \quad \textit{where} \quad c_{hi,kj} = a_{hi} \cdot b_{kj}$$
$$(h=1, 2, ..., m,\ k=1, 2, ..., n).$$

Without loss of generality, all three matrices have the determinant 1. Let $\mu_h^{(m)}$, $\mu_k^{(n)}$, and $\mu_l^{(m,n)}$ be the successive minima of the convex distance functions

$$F^{(m)}(\mathbf{x}) = \max_{h=1,2,...,m} \left| \sum_{i=1}^{m} a_{hi} x_i \right|, \quad F^{(n)}(\mathbf{y}) = \max_{k=1,2,...,n} \left| \sum_{j=1}^{n} b_{kj}\, y_j \right|,$$

and

$$F^{(m,n)}(\mathbf{z}) = \max_{\substack{h=1,2,...,m \\ k=1,2,...,n}} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} c_{hi,kj} z_{ij} \right|,$$

respectively. Denote by $p_l^{(m,n)}$ $(l=1, 2, ..., mn)$ the products

$$\mu_h^{(m)} \mu_k^{(n)}, \quad (h=1, 2, ..., m,\ k=1, 2, ..., n),$$

numbered such that

$$p_1^{(m,n)} \leqslant p_2^{(m,n)} \leqslant \cdots \leqslant p_{mn}^{(m,n)}.$$

Then

$$c_7\, p_l^{(m,n)} \leqslant \mu_l^{(m,n)} \leqslant p_l^{(m,n)}, \qquad (l = 1, 2, \ldots, mn),$$

and in particular,

$$c_7 \mu_1^{(m)} \mu_1^{(n)} \leqslant \mu_1^{(m,n)} \leqslant \mu_1^{(m)} \mu_1^{(n)}, \qquad c_7 \mu_m^{(m)} \mu_n^{(n)} \leqslant \mu_{mn}^{(m,n)} \leqslant \mu_m^{(m)} \mu_n^{(n)}.$$

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