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The Successive Minima in the Geometry of Numbers and the Distinction between Algebraic and Transcendental Numbers

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This paper discusses an application of Minkowski's theory of the successive minima in the geometry of numbers to the problem of the approximation of an algebraic or transcendental number a by algebraic numbers. I consider for simplicity only real numbers a. However, it is obvious that an analogous theory can be established for complex numbers, and also for p-adic numbers, as well as for the field of formal ascending or descending Laurent series with coefficients in an arbitrary field.

1

Let $n \geqslant 2$ be an integer, $\mathbf{R}^n$ the space of all points or vectors $\mathbf{x} = (x_1, \ldots, x_n)$ with real coordinates $x_1, \ldots, x_n$, $\mathbf{0} = (0, \ldots, 0)$ the *origin* of $\mathbf{R}^n$, $a \neq 0$ a real number, and $s \geqslant 2$ a real parameter. Let further L^n be the set of all points $\mathbf{x}$ with integral coordinates; these points are called *lattice points*, and L^n is a lattice. A lattice point $\mathbf{x}$ is said to be *primitive* if the greatest common divisor $\gcd(x_1, \ldots, x_n)$ of its coordinates is equal to 1.

For $\mathbf{x} \in \mathbf{R}^n$ put

$$U(\mathbf{x}) = |x_1 + ax_2 + a^2x_3 + \cdots + a^{n-1}x_n|, \qquad V(\mathbf{x}) = \max(|x_2|, |x_3|, \ldots, |x_n|).$$

We say that $\mathbf{x} \neq \mathbf{0}$ is *singular* if $V(\mathbf{x}) = 0$. There are exactly two primitive singular lattice points, namely

$$\pm\mathbf{e}, \qquad \text{where } \mathbf{e} = (1, 0, 0, \ldots, 0).$$

The maximum

$$F(\mathbf{x}) = \max(s^{n-1}U(\mathbf{x}), s^{-1}V(\mathbf{x}))$$

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is a convex distance function and the point set

$$K\colon F(\mathbf{x}) \leqslant 1$$

is a symmetric convex body in $\mathbf{R}^n$. In fact, K is an n-dimensional parallelepiped with its centre at $\mathbf{0}$ and of volume

$$V(K) = \int \underset{K}{\cdots} \int dx_1 \cdots dx_n = 2^n.$$

Therefore, by Minkowski's theorem on the successive minima [5], there exist n linearly independent primitive lattice points

$$\mathbf{x}^h = (x_{h1}, \ldots, x_{hn}) \qquad (h = 1, 2, \ldots, n),$$

called the generating points, with the following properties:

The determinant $d = \det(x_{hk})_{h,\,k=1,2,\ldots,n}$ *satisfies the inequality* $1 \leqslant |d| \leqslant n!$ (1)

The function values

$$m_h = F(\mathbf{x}^h) \qquad (h = 1, 2, \ldots, n),$$

called the successive minima, satisfy the inequalities

$$0 < m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n, \qquad \frac{1}{n!} \leqslant m_1 m_2 \cdots m_n \leqslant 1. \tag{2}$$

If $X^1, X^2, \cdots, X^n$ *are any* n *linearly independent lattice points numbered such that* $F(\mathbf{X}^1) \leqslant F(\mathbf{X}^2) \leqslant \cdots \leqslant F(\mathbf{X}^n)$, *then*

$$F(\mathbf{X}^h) \geqslant F(\mathbf{x}^h) = m_h \qquad (h = 1, 2, \ldots, n). \tag{3}$$

While the successive minima are unique, each generating point $\mathbf{x}^h$ may be replaced by $-\mathbf{x}^h$, and if two or even more of the minima m_h are equal, there are further possibilities for the lattice points $\mathbf{x}^h$.

2

We want to study the dependence of the successive minima m_h and of the corresponding generating points $\mathbf{x}^h$ on the number $a \neq 0$ when the parameter s is large. The results to be obtained will be different for algebraic a from those for transcendental a.

We first settle the question for which $a \neq 0$ one of the generating lattice points may be singular, say the lattice point $\mathbf{x}^H$.

THEOREM 1. *If there exists a suffix H such that $V(\mathbf{x}^H)=0$ for certain arbitrarily large s, then a is a rational number.*

Proof. Without loss of generality,

$$\mathbf{x}^H = \mathbf{e} = (1, 0, 0, \dots, 0)$$

since $\mathbf{x}^H$ is singular and primitive; therefore

$$U(\mathbf{x}^H) = 1, \qquad V(\mathbf{x}^H) = 0, \qquad m_H = F(\mathbf{e}) = s^{n-1}.$$

There cannot exist a second suffix $h \neq H$ such that also $V(\mathbf{x}^h)=0$ for then $\mathbf{x}^H$ and $\mathbf{x}^h$ would be linearly dependent.

Hence for all suffixes $h \neq H$, $V(\mathbf{x}^h) \neq 0$, hence $V(\mathbf{x}^h) \geqslant 1$, and therefore $m_h = F(\mathbf{x}^h) \geqslant s^{-1}V(\mathbf{x}^h) \geqslant s^{-1}$.

These lower estimates for m_H and m_h imply that

$$1 \geqslant m_1 m_2 \cdots m_n \geqslant s^{n-1}(s^{-1})^{n-1} = 1,$$

hence that

$$m_H = s^{n-1}, \qquad m_h = s^{-1} \qquad \text{for } h \neq H.$$

Here the minima m_h are numbered in order of increasing size. Therefore the suffix H necessarily is equal to n. Since

$$m_h = F(\mathbf{x}^h) = \max(s^{n-1}U(\mathbf{x}^h),\ s^{-1}V(\mathbf{x}^h)),$$

it further follows that

$$U(\mathbf{x}^h) \leqslant s^{-(n-1)} \cdot s^{-1} = s^{-n}, \qquad V(\mathbf{x}^h) = 1 \qquad (h = 1, 2, \dots, n-1).$$

The number a thus satisfies the $n-1$ inequalities:

$$|x_{h1} + a x_{h2} + \cdots + a^{n-1} x_{hn}| \leqslant s^{-n} < 1/2 \qquad (h = 1, 2, \dots, n-1), \tag{4}$$

where

$$V(\mathbf{x}^h) = \max(|x_{h2}|, |x_{h3}|, \dots, |x_{hn}|) = 1 \qquad (h = 1, 2, \dots, n-1). \tag{5}$$

By (5), each of the coordinates x_{hk} $(h = 1, 2, \dots, n-1;\ k = 2, 3, \dots, n)$ can only be equal to either $+1$, -1, or 0. Furthermore, once these $(n-1)^2$ coordinates have been chosen, the remaining coordinates

$$x_{h1} \qquad (h = 1, 2, \dots, n-1)$$

are determined uniquely by the inequalities (4) since a is a constant.

Now let the parameter s tend to infinity. For each such value of s the set of coordinates x_{hk} in (4) has only finitely many possibilities. There exists then an infinite sequence $S=\{s_1, s_2, s_3,...\}$ of distinct values of s tending to infinity such that for all $s_r \in S$ the system of all $n(n-1)$ coordinates x_{hk} in (4) remains fixed. Since $s_r^{-n} \to 0$, it follows that the number a satisfies the system of $n-1$ linear equations

$$x_{h1} + ax_{h2} + \cdots + a^{n-1}x_{hn} = 0 \qquad (h=1, 2,..., n-1) \tag{6}$$

which may be considered as a system of inhomogeneous linear equations for the $n-1$ unknowns $a, a^2,..., a^{n-1}$. It has the determinant

$$D = \begin{vmatrix} x_{12} & x_{13} & \cdots & x_{1n} \\ x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n} \end{vmatrix}.$$

Since $\mathbf{x}^n = \mathbf{e}$, $D = \pm d \neq 0$. Since all x_{hk} in (4) are rational integers, the assertion follows at once from Cramer's rule.

COROLLARY. *The denominator of a cannot exceed $\sqrt{n-1}$.*

Proof. Since all elements of D are $+1$, -1, or 0, it is well known that

$$|D| \leqslant (n-1)^{(n-1)/2}.$$

By Cramer's formula, a^{n-1} has then a denominator not greater than $(n-1)^{(n-1)/2}$, and hence the denominator of a cannot be greater than $\sqrt{n-1}$.

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From now on let a be irrational. Theorem 1 implies then that for all sufficiently large s

$$V(\mathbf{x}^h) \geqslant 1 \qquad (h=1, 2,..., n).$$

Thus from the definition of $F(\mathbf{x})$,

$$U(\mathbf{x}^h) \leqslant s^{-(n-1)} \cdot m_h, \qquad 1 \leqslant V(\mathbf{x}^h) \leqslant s \cdot m_h \qquad (h=1, 2,..., n),$$

so that on eliminating the parameter s,

$$|x_{h1} + ax_{h2} + \cdots + a^{n-1}x_{hn}| \leqslant m_h^n (\max(|x_{h2}|, |x_{h3}|,..., |x_{hn}|))^{-(n-1)} \tag{7}$$

for $h=1, 2,..., n$.

This is a system of n linearly independent approximation polynomials for a. Here the right-hand sides are small only if the succesive minima m_h are not too large while $\max(|x_{h2}|, |x_{h3}|, \ldots, |x_{hn}|)$ is sufficiently big. In fact, this maximum may stay bounded if the left-hand side of (7) can vanish, i.e., if a is algebraic of degree at most $n-1$.

By the inequalities (2),

$$m_1 \leqslant 1, \qquad m_n \geqslant (n!)^{-1/n},$$

because

$$m_1^n \leqslant m_1 m_2 \cdots m_n \leqslant m_n^n.$$

When m_1 is very small, m_n necessarily is very large. As the later estimates for the m_n will show, this can in fact happen.

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The proof of Theorem 1 can be generalised and then implies the following result.

THEOREM 2. *Denote by N an integer such that $1 \leqslant N \leqslant n-1$, and by $c_1 > 0$ a constant which does not depend on s. Assume that there exists an infinite sequence $S = \{s_1, s_2, s_3, \ldots\}$ of numbers $s \geqslant 2$ tending to infinity such that simultaneously*

$$m_h \leqslant c_1 s^{-1} \qquad (h = 1, 2, \ldots, n-N)$$

for all $s \in S$. Then a is algebraic and at most of degree N.

Proof. The assertion is certainly true if a is rational. Assume then that a is irrational and hence by Theorem 1,

$$V(\mathbf{x}^h) \geqslant 1 \qquad (h = 1, 2, \ldots, n).$$

For all $s \in S$ by the hypothesis,

$$m_h = F(\mathbf{x}^h) = \max(s^{n-1} U(\mathbf{x}^h), s^{-1} V(\mathbf{x}^h)) \leqslant c_1 s^{-1} \qquad (h = 1, 2, \ldots, n-N)$$

and therefore

$$U(\mathbf{x}^h) \leqslant c_1 s^{-n} \qquad \text{and} \qquad V(\mathbf{x}^h) \leqslant c_1 \qquad (h = 1, 2, \ldots, n-N),$$

or in explicit form,

$$|x_{h1}+ax_{h2}+\cdots+a^{n-1}x_{hn}|\leqslant c_1 s^{-n}$$
$$\max(|x_{h2}|, |x_{h3}|,\ldots, |x_{hn}|)\leqslant c_1 \qquad (h=1, 2,\ldots, n-N). \tag{8}$$

Let now $s\in S$ be already so large that

$$c_1 s^{-n}<1/2.$$

The first inequalities (8) determine then the coordinates x_{h1} uniquely in terms of the coordinates x_{hk} where $k\geqslant 2$, while the second inequalities (8) show that the matrix

$$\mathbf{X}=\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n-N,1} & x_{n-N,2} & \cdots & x_{n-N,n} \end{pmatrix}$$

consists of bounded integers and so has only finitely many possibilities. Moreover, since $\mathbf{x}^1, \mathbf{x}^2,\ldots, \mathbf{x}^{n-N}$ are linearly independent, $\mathbf{X}$ has the exact rank $n-N$. This matrix will of course vary for different $s\in S$. It is, however, clear that $\mathbf{X}$ remains fixed when s runs over a suitable infinite subsequence S^* of S. As s runs over S^*, s tends to infinity and hence $c_1 s^{-n}$ tends to zero.

Hence the first inequalities (8) imply the equations

$$x_{h1}+ax_{h2}+\cdots+a^{n-1}x_{hn}=0 \qquad (h=1, 2,\ldots, n-N).$$

Denote by $g_1, g_2,\ldots, g_{n-N}$ a set of $n-N$ integers not all zero and put

$$G_i=\sum_{h=1}^{n-N} g_h x_{hi} \qquad (i=1, 2,\ldots, n),$$

so that

$$\sum_{h=1}^{n-N} g_h(x_{h1}+ax_{h2}+\cdots+a^{n-1}x_{hn})=G_1+aG_2+\cdots+a^{n-1}G_n=0.$$

Since $\mathbf{X}$ has the rank $n-N\geqslant 1$, the sums $G_1, G_2,\ldots, G_n$ cannot all vanish, and all these sums are integers since the coefficients g_i are so.

We now choose the integers g_i such that the $n-N-1$ homogeneous linear equations

$$G_{N+2}=G_{N+3}=\cdots=G_n=0$$

are satisfied, while $G_1, G_2, \dots, G_{N+1}$ are not all zero. The number a is now a root of the algebraic equation

$$G_1 + aG_2 + \cdots + a^N G_{N+1} = 0$$

with integral coefficients. Since it is not possible that only G_1 is distinct from 0, the assertion follows at once.

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Denote from now by $c_2, c_3, \dots,$ positive constants which do not depend on s, but may depend on a and n.

If a is an algebraic number, lower and upper estimates for the successive minima m_h are as follows:

THEOREM 3. *Let $a \neq 0$ be any real algebraic number, say of the exact degree N, and let the parameter s be already sufficiently large. If $1 \leqslant N \leqslant n$, then*

$$s^{-1} \leqslant m_h \leqslant c_2 s^{-1} \qquad \text{for } h = 1, 2, \dots, n-N,$$

$$c_3 s^{(n-N)/N} \leqslant m_h \leqslant c_4 s^{(n-N)/N} \qquad \text{for } h = n-N+1, n-N+2, \dots, n.$$

If, however, $N > n$ and if ε is an arbitrarily small positive number, then for s greater than a number depending on ε

$$s^{-\varepsilon} \leqslant m_h \leqslant s^{+\varepsilon} \qquad \text{for} \quad h = 1, 2, \dots, n.$$

Proof. A general lattice point $\mathbf{x} \neq \mathbf{0}$ is said to be of class A if

$$U(\mathbf{x}) \neq 0$$

and of class B if

$$U(\mathbf{x}) = 0.$$

If $N = n$, evidently all lattice points $\mathbf{x} \neq 0$ are of class A; this is true thus in particular for the n lattice points $\mathbf{x}^h$.

Next let $1 \leqslant N \leqslant n-1$. The algebraic number $a \neq 0$ satisfies an irreducible and primitive algebraic equation of degree $N \leqslant n-1$ with integral coefficients, say the equation

$$q_1 + aq_2 + \cdots + a^N q_{N+1} = 0, \qquad \text{where } q_1 \neq 0 \text{ and } q_{N+1} \neq 0.$$

The corresponding $n-N$ lattice points

$$\mathbf{X}^1 = (q_1, q_2, \ldots, q_{N+1}, 0, \ldots, 0), \mathbf{X}^2 = (0, q_1, q_2, \ldots, q_{N+1}, 0, \ldots, 0), \ldots,$$
$$\mathbf{X}^{n-N} = (0, \ldots, 0, q_1, q_2, \ldots, q_{n-N})$$

evidently are linearly independent and satisfy the relations

$$U(\mathbf{X}^h) = 0, \qquad V(\mathbf{X}^h) = c_2 \qquad \text{for } h = 1, 2, \ldots, n-N,$$

where

$$c_2 = \max(|q_1|, |q_2|, \ldots, |q_{N+1}|).$$

The points $\mathbf{X}^h$ are therefore of class B. *There cannot exist any further lattice point* $\mathbf{x}$ *of class* B *which is linearly independent of* $\mathbf{X}^1, \ldots, \mathbf{X}^{n-N}$. For otherwise there are integers $g \neq 0$, $g_1, \ldots, g_{n-N}$ such that the lattice point

$$\mathbf{X} = g\mathbf{x} + g_1 \mathbf{X}^1 + \cdots + g_{n-N} \mathbf{X}^{n-N} = (X_1, X_2, \ldots, X_n),$$

say, satisfies the linear equations

$$X_{N+1} = X_{N+2} = \cdots = X_n = 0,$$

while $X_1, X_2, \ldots, X_N$ are not all zero. However, also $\mathbf{X}$ is of class B and therefore

$$U(\mathbf{X}) = X_1 + aX_2 + \cdots + a^{N-1} X_N = 0.$$

Thus a satisfies an algebraic equation with integral coefficients at most of degree $N-1$, contrary to the hypothesis.

By the definition of the lattice points $\mathbf{X}^h$,

$$F(\mathbf{X}^h) = s^{-1} V(\mathbf{X}^h) = c_2 s^{-1} \qquad (h = 1, 2, \ldots, n-N).$$

It follows then from the property (3) of the successive minima that

$$m_h \leqslant F(\mathbf{X}^h) = c_2 s^{-1} \qquad (h = 1, 2, \ldots, n-N). \tag{9}$$

To this we may add the lower estimates

$$m_h \geqslant s^{-1} \qquad (h = 1, 2, \ldots, n-N), \tag{10}$$

because for all suffixes $h = 1, 2, \ldots, n-N$, $V(\mathbf{x}^h) \neq 0$, hence $V(\mathbf{x}^h) \geqslant 1$ and $F(\mathbf{x}^h) \geqslant s^{-1} V(\mathbf{x}^h)$.

From these estimates,

$$s^{-(n-N)} \leqslant m_1 m_2 \cdots m_{n-N} \leqslant c_2^{n-N} s^{-(n-N)},$$

hence by Minkowski's inequality (2),

$$\frac{1}{n!}\, c_2^{-(n-N)}\, s^{n-N} \leqslant m_{n-N+1} m_{n-N+2} \cdots m_n \leqslant s^{n-N}. \tag{11}$$

We note that in the special case when $N=1$ these formulae show already that

$$s^{-1} \leqslant m_h \leqslant c_2 s^{-1} \text{ for } h=1, 2,..., n-1;\ (1/n!)\, c_2^{-(n-1)}\, s^{n-1} \leqslant m_n \leqslant s^{n-1},$$

which is the assertion.

Now assume that $2 \leqslant N \leqslant n$. By a classical method based on considering the norm $N(x_1 + ax_2 + \cdots + a^{n-1}x_n)$, where $\mathbf{x} \in L^n$ it can be proved that

There exists a constant $C>0$ depending only on a and n such that for all lattice points $\mathbf{x} \in L^n$

$$U(\mathbf{x}) \geqslant C^N V(\mathbf{x})^{-(N-1)} \textit{ if } U(\mathbf{x}) \neq 0 \textit{ and } V(\mathbf{x}) \neq 0.$$

This estimate may in particular be applied to all the lattice points $\mathbf{x}^h$ for which $U(\mathbf{x}^h) \neq 0$; for the second condition $V(\mathbf{x}^h) \neq 0$ holds by Theorem 1. Thus for these lattice points,

$$m_h = F(\mathbf{x}^h) \geqslant \max(s^{n-1} \cdot C^N V(\mathbf{x}^h)^{-(N-1)}, s^{-1} V(\mathbf{x}^h)).$$

If here

$$V(\mathbf{x}^h) = Cs^{n/N},$$

then both terms under the maximum sign are equal to

$$Cs^{(n-N)/N};$$

otherwise one of the two terms is greater. We obtain then the result that

$$m_h = F(\mathbf{x}^h) \geqslant Cs^{(n-N)/N} \qquad \text{if } U(\mathbf{x}^h) \neq 0.$$

Assume now that s is already so large that

$$c_2 s^{-1} < Cs^{(n-N)/N}.$$

What has been proved so far implies then that

$$m_h \geqslant Cs^{(n-N)/N} \qquad \text{for } h = n-N+1, n-N+2,..., n. \tag{12}$$

If this lower estimate is substituted for all but one of the factors m_h in the equality (11), we further obtain the upper estimates

$$m_h \leqslant (Cs^{(n-N)/N})^{1-N} \cdot s^{n-N} = C^{1-N} s^{(n-N)/N}. \tag{13}$$

On combining the estimates (9), (10), (12), and (13), we obtain the assertion of the theorem when $1 \leqslant N \leqslant n$.

We note that in the special case when $N = n$,

$$c_3 \leqslant m_h \leqslant c_4 \qquad \text{for } h = 1, 2, \ldots, n.$$

Consider finally the case when the degree N of a is greater than n. Now the elementary method used so far is no longer powerful enough and we must apply the following deep theorem by Schmidt; I refer for convenience to his book [6]:

If a is an algebraic number of degree $N \geqslant n+1$ and ε is an arbitrarily small positive constant, then there exists a positive constant $c(\varepsilon)$ such that

$$U(\mathbf{x}) \geqslant c(\varepsilon) V(\mathbf{x})^{-(n-1+\varepsilon)} \qquad \textit{if } \mathbf{x} \in L^n \textit{ and } V(\mathbf{x}) \neq 0.$$

This theorem may be applied in particular to all the lattice points $\mathbf{x}^h$ because $V(\mathbf{x}^h) \neq 0$ for $h = 1, 2, \ldots, n$ by Theorem 1. It follows that for all h,

$$m_h = F(\mathbf{x}^h) \geqslant \max(s^{n-1} \cdot c(\varepsilon) V(\mathbf{x}^h)^{-(n-1+\varepsilon)}, s^{-1} V(\mathbf{x}^h)).$$

If here

$$V(\mathbf{x}^h)^{n+\varepsilon} = c(\varepsilon)\, s^n,$$

then both expressions under the maximum sign assume the same value

$$c(\varepsilon)^{1/(n+\varepsilon)} s^{-\varepsilon/(n+\varepsilon)};$$

otherwise one of the two terms is greater.

It follows then that

$$m_h \geqslant c(\varepsilon)^{1/(n+\varepsilon)} s^{-\varepsilon/(n+\varepsilon)} \qquad (h = 1, 2, \ldots, n).$$

On substituting again this lower estimate for $n-1$ factors in Minkowski's inquality $m_1 m_2 \cdots m_2 \leqslant 1$, we further obtain the upper estimates

$$m_h \leqslant c(\varepsilon)^{-(n-1)/(n+\varepsilon)} s^{\varepsilon(n-1)/(n+\varepsilon)} \qquad (h = 1, 2, \ldots, n).$$

Here finally let s be sufficiently large. The lower and upper estimate combine then to the result that

$$s^{-\varepsilon} \leqslant m_h \leqslant s^{+\varepsilon} \qquad (h = 1, 2,..., n),$$

as was to be proved.

6

Theorem 3 establishes estimates for the successive minima m_h in all cases when **a** is algebraic. No such general results can be given when **a** is transcendental. We can, however, state several results which explain how characteristic the upper estimates are for m_h in Theorems 2 and 3 for algebraic numbers.

By Theorem 2 the number a is algebraic if there exist a positive number c_1, an integer N with $1 \leqslant N \leqslant n-1$, and an infinite sequence S of positive numbers $s \geqslant 2$ tending to infinity such that

$$m_h \leqslant c_1 s^{-1} \qquad (h = 1, 2,..., n-N).$$

As will now be proved, here the upper bound $c_1 s^{-1}$ cannot be replaced by any larger function of s.

THEOREM 4. *Let $T(s) > 0$ be any function of $s \geqslant 2$ such that*

$$\lim_{s \to \infty} T(s) = \infty.$$

Then there exist a real transcendental number a and an infinite sequence S of numbers $s \geqslant 2$ tending to infinity such that

$$m_h \leqslant T(s)\, s^{-1} \qquad \text{for} \quad s \in S \quad (h = 1, 2,..., n-1).$$

Proof. Define two sequences of positive integers e_r and g_r, where

$$e_1 = 2 \qquad \text{and} \qquad g_r = e_1 e_2 \cdots e_r$$

by the recursive condition that if $e_1, e_2,..., e_r$ and hence also $g_1, g_2,..., g_r$ have already been fixed, then e_{r+1} is to be the smallest integer greater than e_r for which

$$T(2^{g_{r+1/n}}) \geqslant 2^{g_r + 1} \qquad (r = 1, 2, 3,...). \tag{14}$$

Such an integer exists because $T(s)$ may by hypothesis assume arbitrarily large values.

Next take for a the infinite series

$$a=\sum_{r=1}^{\infty} 2^{-g_r}.$$

which converges and lies in the interval $0<a<1$. Further put for all r

$$q_{1r}=-2^{g_r}\sum_{j=1}^{r} 2^{-g_j}, \qquad q_{2r}=2^{g_r}, \qquad R_r=q_{1r}+aq_{2r}.$$

Then q_{1r} and q_{2r} are integers satisfying

$$0<-q_{1r}<q_{2r}.$$

Further

$$R_r=2^{g_r-g_{r+1}}+2^{g_r-g_{r+2}}+2^{g_r-g_{r+3}}+\cdots,$$

so that

$$R_r=\rho_r 2^{g_r-g_{r+1}}, \qquad \text{where } 1<\rho_r<2. \tag{15}$$

Since $g_{r+1}=e_{r+1}g_r$ is for large r an arbitrarily large multiple of g_r, the formulae for q_{2r} and R_r show that a is a Liouville number, hence is transcendental.

Now for $r=1, 2, 3,\ldots$, form the $n-1$ lattice points in L^n,

$$\mathbf{X}^{1r}=(q_{1r}, q_{2r}, 0,\ldots, 0), \qquad \mathbf{X}^{2r}=(0, q_{1r}, q_{2r}, 0,\ldots, 0),\ldots,$$
$$\mathbf{X}^{n-1,r}=(0,\ldots, 0, q_{1r}, q_{2r}).$$

It is clear that these points are linearly independent and that

$$U(\mathbf{X}^{hr})=a^{h-1}R_r, \qquad V(\mathbf{X}^{hr})=q_{2r} \qquad (h=1, 2,\ldots, n-1),$$

hence also

$$F(\mathbf{X}^{hr})=\max(s^{n-1}.a^{h-1}R_r, s^{-1}q_{2r}) \qquad (h=1, 2,\ldots, n-1).$$

Here $0<a<1$. Hence by (15) and by the definition of q_{2r},

$$F(\mathbf{X}^{hr})\leqslant 2^{g_r+1}\cdot\max(s^{n-1}\cdot 2^{-g_{r+1}}, s^{-1}) \qquad (h=1, 2,\ldots, n-1).$$

For each suffix $r=1, 2, 3,\ldots$, now let s_r be the number

$$s_r=2^{g_{r+1/n}},$$

and let S be the set of all s_r which are at least 2. By (14),

$$T(s_r) \geqslant 2^{g_r+1},$$

whence it follows that

$$F(\mathbf{X}^{hr}) \leqslant T(s_r)\, s_r^{-1} \qquad (h = 1, 2, ..., n-1).$$

Finally apply once more the property (3) of the successive minima. It follows then that

$$m_h \leqslant T(s)\, s^{-1} \qquad \text{for } s \in S \quad (h = 1, 2, ..., n-1),$$

as was to be proved.

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When a is algebraic of degree $N = n$, Theorem 3 gave the estimates

$$c_3 \leqslant m_h \leqslant c_4 \qquad (h = 1, 2, ..., n). \tag{16}$$

If further a is algebraic of degree $N > n$, then we deduced from Schmidt's theorem that for every $\varepsilon > 0$ and for all sufficiently large s,

$$s^{-\varepsilon} \leqslant m_h \leqslant s^{+\varepsilon} \qquad (h = 1, 2, ..., n). \tag{17}$$

Neither of these results is characteristic of algebraic numbers.

In the case of (16), theorems by Cassels [1] and by Davenport [2, 3] imply that there are non-countably many real numbers a with this property if c_3 and c_4 are suitably chosen positive constants. There are thus also transcendental numbers with this property.

Next, a beautiful theorem by Sprindžuk [7] shows that almost all real numbers a have the property (17) for sufficiently large s however small the number $\varepsilon > 0$ is chosen. In particular, almost all real transcendental numbers a satisfy (17).

Using my classification of transcendental numbers divided into the three classes S, T, and U (see, e.g., [4]) it is further easy to show the following result:

> *If a is a real S-number, then there exists a number δ satisfying $0 < \delta < 1$ which is independent of n and s such that for all sufficiently large s,*

$$s^{-1+\delta} \leqslant m_h \leqslant s^{(n-1)(1-\delta)} \qquad (h = 1, 2, ..., n). \tag{18}$$

If a is a real T-number, then there still exists a number δ with the property (18), *but this number now depends on n and tends to zero as n tends to infinity. It is, however, independent of s. If finally, a is a real U-number, then there is no constant δ with the property* (18) *which is independent of s.*

By the way of example, if $\omega \neq 0$ is any real algebraic number, then $a = e^{\omega}$ is an S-number, while both $\log 2$ and π are either S-numbers or T-numbers.

In the special case of $a = e$, an old result of mine [4] enables one to show the following very sharp estimate:

There exists an absolute constant $C > 0$ such that for all sufficiently large s and for all $n \geqslant 2$,

$$s^{-C \cdot n \log n / \log \log s} \leqslant m_h \leqslant s^{+C \cdot n \log n / \log \log s} \qquad (h = 1, 2, \ldots, n).$$

This estimate is thus stronger than (17).

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